Geometric Distribution

This describes a game with two outcomes, ‘win’ or ‘lose’. The game is played up to and including the first ‘win’, e.g. as in a driving test.

If the probability of a win is \( p \), and \( X \) is the number of trials up to and including the first ‘win’, we say \( X \sim \text{Geo}(p) \).

e.g. Two in three driving tests are failures. Find the probability that a person passes on their

a/ 2nd attempt \( X \sim \text{Geo}(\frac{1}{3}) \) \( P(X=2) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} \)

b/ 4th attempt \( P(X=4) = (\frac{2}{3})^3 \times (\frac{1}{3}) = \frac{8}{81} \)

c/ 8th attempt \( P(X=8) = (\frac{2}{3})^7 \times (\frac{1}{3}) = \frac{128}{6561} \)

Summary

For a game with two outcomes, win or lose, the probability of the first win occurring on turn \( n \), ‘\( n \)’ is,

\[ P(X \geq n) = (1-p)^{n-1} \cdot p \quad \text{where} \quad X \sim \text{Geo}(p) \]

e.g. For \( X \sim \text{Geo}(0.7) \)

a/ \( P(X=8) = 0.3^7 \times 0.7 = 1.53 \times 10^{-4} \)

b/ \( P(X > 5) = 0.3^5 = 2.43 \times 10^{-3} \)

The ‘bet’ can be said to be won as soon as the 5th loss has occurred. In the geometric distribution ‘\( > \)’ probabilities are the easiest to work out.

c/ \( P(X > 3) = P(X > 2) = 0.3^2 = 0.09 \)
\( P(X \leq 10) = 1 - P(X > 10) \)
\( = 1 - 0.3^{10} = 0.9999940851 \)
\( \approx 1.00 \) (3sf)

Do want 10, 9, 8, ...
Don't want 11, 12, 13, ...

\( P(5 \leq X \leq 7) \)
Do want 5, 6

\( P(X > 4) - P(X > 6) = 0.3^4 - 0.3^6 = 7.37 \times 10^{-3} \)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( P(X = x) )</th>
<th>( E(X) )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial ( B(n, p) )</td>
<td>( \binom{n}{x} p^x (1-p)^{n-x} )</td>
<td>np</td>
<td>np(1-p)</td>
</tr>
<tr>
<td>Uniform distribution over 1, 2, ..., ( n ) ( U(n) )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{n+1}{2} )</td>
<td>( \frac{1}{12}(n^2 - 1) )</td>
</tr>
<tr>
<td>Geometric distribution ( \text{Geo}(p) )</td>
<td>( (1-p)^{x-1} p )</td>
<td>( \frac{1}{p} )</td>
<td>( \frac{1-p}{p^2} )</td>
</tr>
<tr>
<td>Poisson ( \text{Po}(\lambda) )</td>
<td>( e^{-\lambda} \frac{\lambda^x}{x!} )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
</tbody>
</table>

\[ E(X) = \frac{1}{p} \]
\[ \text{Var} = \frac{1-p}{p^2} \]

**Assumptions for a Geometric Distribution to be Valid**

There must be modified to fit the context of the situation:

- Trials are independent
- Fixed probability of success
\( i/ \quad X \sim \text{geo}(\frac{2}{5}) \)

\( a/ \quad P(X = 5) = \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right) = \frac{162}{3125} \)

\( b/ \quad P(X < 5) = 1 - P(X > 4) \)

\[ = 1 - \left(\frac{3}{5}\right)^4 = \frac{544}{625} \quad \text{Do want 4, 3, 2, ...} \]

\[ \text{Don’t want 5, 6, 7, 8, ...} \]

\( ii/a/ \quad X \sim \text{B}(5, \frac{2}{5}) \), \quad P(X = 1) = 0.2592 \)

\( b/ \quad \text{1 goal in 5 shots} \)

\[ 0.2592 \times \frac{2}{5} = \frac{324}{3125} \]

\( i/\quad X \sim \text{geo}\left(\frac{1}{8}\right) \quad P(X = 3) = \left(\frac{7}{8}\right)^3 \left(\frac{1}{8}\right) = \frac{49}{512} \)

\( b/ \quad P(X > 3) = \left(\frac{7}{8}\right)^3 = \frac{343}{512} \)

\( ii/ \quad E(X) = \frac{1}{\left(\frac{1}{8}\right)} = 8 \)

\( iii/ \quad X \sim \text{B}(15, \frac{1}{8}) \quad P(X = 2) = 0.289 \)
The random variable $X$ has the distribution $\text{Geo}(0.2)$. Find

(i) $P(X = 3)$.
(ii) $P(3 \leq X \leq 5)$,
(iii) $P(X > 4)$.

Two independent values of $X$ are found.

(iv) Find the probability that the total of these two values is 3.

\[
P(X = 3) = 0.8^2 \times 0.2 = \frac{16}{125}
\]
\[
P(X > 2) - P(X > 5) = 0.8^2 - 0.8^5 \quad \text{Want 3, 4, 5}
\]
\[
= \frac{64}{125}
\]
\[
P(X > 4) = 0.8^4 = \frac{256}{625}
\]

\[
X_1 \quad X_2 \quad P(X_1) \quad P(X_2)
\]
\[
1 \quad 2 \quad 0.2 \quad 0.2 \times 0.8 = 0.032
\]
\[
2 \quad 1 \quad 0.2 \times 0.8 \quad 0.2
\]
\[
= 0.064
\]

Q13, (Jun 2016, Q7)
On average Marie scores a goal on 20% of her shots. The variable random $X$ is the number of shots Marie takes, up to and including her first goal.

(i) State two conditions needed for $X$ to have a geometric distribution.

(ii) Assuming these conditions are satisfied, find the probability that

(a) $X = 3$,

(b) $X < 10$,

(c) $9 < X < 20$. 

\[
P(X = 3) = 0.8^2 \times 0.2 = \frac{16}{125}
\]
\[
1 - P(X > 9) = 1 - 0.8^9 = 0.866
\]
\[
P(X > 9) - P(X > 19) = 0.8^9 - 0.8^{19} = 0.120.
\]