Proof By Induction (Inequalities)

e.g. Prove $2^n > 1 + n$ for all $n \geq 2$, $n \in \mathbb{N}$

1. Prove for the base case
   
   Let $n = 2 \implies LHS = 2^2 = 4$
   
   $RHS = 1 + 2 = 3$
   
   $LHS > RHS \therefore$ true for $n = 2$

2. Assume true for $n = k$
   
   Let $n = k$ and assume true
   
   i.e. $2^k > 1 + k$

3. Let $n = k + 1$ and prove true using assumption
   
   At this point look at what we need to do to tie the LHS of our assumption to get the LHS of our target

   Assuming $2^k > 1 + k$
   
   $\implies 2(2^k) > 2(1 + k)$ (Multiplying both sides of our assumption by 2 gives another statement we can assume true)
   
   $\implies 2^{k+1} > 2 + 2k$
   
   $\implies 2^{k+1} > (k + 2) + k > k + 2$ Since $k > 0$ for $k > 1$
   
   $\therefore 2^{k+1} > k + 2$

4. Conclude
   
   If true for $n = k$ then true for $n = k + 1$
   
   Since true for $n = 2$ then true for all integers $n \geq 2$
e.g. Prove \( 2^n > 2n \) for \( n \geq 3, n \in \mathbb{N} \)

1. **Base case**
   - Let \( n = 3 \)
   - \( \text{LHS} = 2^3 = 8 \)
   - \( \text{RHS} = 2(3) = 6 \)
   - \( \text{LHS} > \text{RHS} \implies \text{true for } n = 3 \)

2. **Assumption**
   - Let \( n = k \) and assume true
   - i.e. \( 2^n > 2k \)

3. **Inductive Step**
   - Assuming \( 2^k > 2k \)
   - \( \implies 2(2^k) > 2(2k) \) \( \leftarrow \) Multiply the LHS of our assumption by 2 to get the LHS of our target
   - \( \implies 2^{k+1} > 4k \)
   - \( \implies 2^{k+1} > (2k+2)+(2k-2) > 2k+2 \) \( \text{since } 2k-2 > 0 \) for \( k \geq 2 \)
   - \( \text{Must be greater than above line to be true} \)
   - \( \implies 2^{k+1} > 2k+2 \)

4. **Conclusion**
   - If true for \( n = k \) then true for \( n = k+1 \)
   - Since true for \( n = 3 \) then true for all integers \( n \geq 3 \)
e.g. Prove that \( n! > 2^n \) for all \( n \geq 4 \), \( n \in \mathbb{N} \)

1) Base case

Let \( n = 4 \) \( \Rightarrow \) LHS = \( 4! = 24 \)
RHS = \( 2^4 = 16 \)
LHS > RHS \( \therefore \) true for \( n = 4 \)

2) Assumption

Let \( n = k \) and assume true
i.e. \( k! > 2^k \)

3) Inductive step

Assuming \( k! > 2^k \)

\[(k+1)! > (k+1)2^k \]

\[\Rightarrow (k+1)! > \frac{2 \times 2^k}{k+1} \]

Multiply both sides by \( k+1 \) to get LHS of target

\[\text{since } k+1 \geq 2 \text{ when } k \geq 2 \]

We have made RHS smaller so \( (k+1)! > 2^{k+1} \)

\[\Rightarrow \]

4) Conclusion

If true for \( n = k \) then true for \( n = k+1 \)

Since true for \( n = 4 \) then true for all integers \( n \geq 4 \)