

Integrating Parametric Functions (From Edexcel 6666)

Q1, (Jan 2006, Q8)

(a)	Solves $y = 0 \Rightarrow \cos t = \frac{1}{2}$ to obtain $t = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ (need both for A1)	M1 A1	
	Or substitutes both values of t and shows that $y = 0$		(2)
(b)	$\frac{dx}{dt} = 1 - 2 \cos t$	M1 A1	
	$\text{Area} = \int y dx = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2 \cos t)(1 - 2 \cos t) dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2 \cos t)^2 dt \quad * \quad \text{AG}$	B1	(3)
(c)	$\text{Area} = \int 1 - 4 \cos t + 4 \cos^2 t dt \quad \text{3 terms}$	M1	
	$= \int 1 - 4 \cos t + 2(\cos 2t + 1) dt \quad (\text{use of correct double angle formula})$	M1	
	$= \int 3 - 4 \cos t + 2 \cos 2t dt$		
	$= [3t - 4 \sin t + \sin 2t]$	M1 A1	
	Substitutes the two correct limits $t = \frac{5\pi}{3}$ and $\frac{\pi}{3}$ and subtracts.	M1	
	$= 4\pi + 3\sqrt{3}$	A1A1	(7)
			[12]

Q2, (Jan 2008, Q7)

<p>(a)</p> $\left[x = \ln(t+2), y = \frac{1}{t+1} \right], \Rightarrow \frac{dx}{dt} = \frac{1}{t+2}$ $\text{Area}(R) = \int_{\ln 2}^{\ln 4} \frac{1}{t+1} dx = \int_0^2 \left(\frac{1}{t+1} \right) \left(\frac{1}{t+2} \right) dt$ <p>Changing limits, when: $x = \ln 2 \Rightarrow \ln 2 = \ln(t+2) \Rightarrow 2 = t+2 \Rightarrow t = 0$ $x = \ln 4 \Rightarrow \ln 4 = \ln(t+2) \Rightarrow 4 = t+2 \Rightarrow t = 2$</p> <p>Hence, $\text{Area}(R) = \int_0^2 \frac{1}{(t+1)(t+2)} dt$</p>	<p>Must state $\frac{dx}{dt} = \frac{1}{t+2}$</p> <p>Area = $\int \frac{1}{t+1} dx$. Ignore limits.</p> <p>$\int \left(\frac{1}{t+1} \right) \times \left(\frac{1}{t+2} \right) dt$. Ignore limits.</p> <p>changes limits $x \rightarrow t$ so that $\ln 2 \rightarrow 0$ and $\ln 4 \rightarrow 2$</p>	<p>B1</p> <p>M1;</p> <p>A1 AG</p> <p>B1</p>
<p>b)</p> $\left(\frac{1}{(t+1)(t+2)} \right) = \frac{A}{t+1} + \frac{B}{t+2}$ <p>$1 = A(t+2) + B(t+1)$</p> <p>Let $t = -1, 1 = A(1) \Rightarrow \underline{A = 1}$</p> <p>Let $t = -2, 1 = B(-1) \Rightarrow \underline{B = -1}$</p> $\int_0^2 \frac{1}{(t+1)(t+2)} dt = \int_0^2 \frac{1}{t+1} - \frac{1}{t+2} dt$ $= [\ln(t+1) - \ln(t+2)]_0^2$ $= (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$ $= \ln 3 - \ln 4 + \ln 2 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right)$	<p>$\frac{A}{t+1} + \frac{B}{t+2}$ with A and B found</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <p>Finds both A and B correctly. Can be implied. (See note below)</p> </div> <p>Either $\pm a \ln(t+1)$ or $\pm b \ln(t+2)$ Both \ln terms correctly ft.</p> <p>Substitutes both limits of 2 and 0 and subtracts the correct way round.</p> <p>$\frac{\ln 3 - \ln 4 + \ln 2}{}$ or $\ln\left(\frac{3}{4}\right) - \ln\left(\frac{1}{2}\right)$ or $\ln 3 - \ln 2$ or $\ln\left(\frac{3}{2}\right)$ (must deal with $\ln 1$)</p>	<p>M1</p> <p>A1</p> <p>dM1 A1 $\sqrt{\quad}$</p> <p>ddM1</p> <p>A1 aef isw</p>

[4]

[6]

Takes out brackets.

Writing down $\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} + \frac{1}{t+2}$ means first M1A0 in (b).

Writing down $\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} - \frac{1}{t+2}$ means first M1A1 in (b).

	$x = \ln(t+2), \quad y = \frac{1}{t+1}$		
(c)	$e^x = t+2 \Rightarrow t = e^x - 2$	Attempt to make $t = \dots$ the subject giving $t = e^x - 2$	M1 A1
	$y = \frac{1}{e^x - 2 + 1} \Rightarrow y = \frac{1}{e^x - 1}$	Eliminates t by substituting in y giving $y = \frac{1}{e^x - 1}$	dM1 A1
Aliter (c) Way 2	$t+1 = \frac{1}{y} \Rightarrow t = \frac{1}{y} - 1$ or $t = \frac{1-y}{y}$	Attempt to make $t = \dots$ the subject	M1
	$y(t+1) = 1 \Rightarrow yt + y = 1 \Rightarrow yt = 1 - y \Rightarrow t = \frac{1-y}{y}$	Giving either $t = \frac{1}{y} - 1$ or $t = \frac{1-y}{y}$	A1
	$x = \ln\left(\frac{1}{y} - 1 + 2\right)$ or $x = \ln\left(\frac{1-y}{y} + 2\right)$	Eliminates t by substituting in x	dM1
	$x = \ln\left(\frac{1}{y} + 1\right)$		
	$e^x = \frac{1}{y} + 1$		
	$e^x - 1 = \frac{1}{y}$		
	$y = \frac{1}{e^x - 1}$	giving $y = \frac{1}{e^x - 1}$	A1
(d)	Domain: $x > 0$	$x > 0$ or just > 0	B1
			[4] [1]
			15 marks

Q3, (Jun 2008, Q8)

<p>(a) At $P(4, 2\sqrt{3})$ either $4 = 8\cos t$ or $2\sqrt{3} = 4\sin 2t$</p> <p>\Rightarrow only solution is $t = \frac{\pi}{3}$ where $0 \leq t \leq \frac{\pi}{2}$</p>	<p>$4 = 8\cos t$ or $2\sqrt{3} = 4\sin 2t$</p> <p>$t = \frac{\pi}{3}$ or awrt 1.05 (radians) only stated in the range $0 \leq t \leq \frac{\pi}{2}$</p>	<p>M1</p> <p>A1</p>
<p>(b) $x = 8\cos t$, $y = 4\sin 2t$</p> <p>$\frac{dx}{dt} = -8\sin t$, $\frac{dy}{dt} = 8\cos 2t$</p> <p>At P, $\frac{dy}{dx} = \frac{8\cos(\frac{2\pi}{3})}{-8\sin(\frac{\pi}{3})}$</p> <p>$\left\{ = \frac{8(-\frac{1}{2})}{(-8)(\frac{\sqrt{3}}{2})} = \frac{1}{\sqrt{3}} = \text{awrt } 0.58 \right\}$</p> <p>Hence $m(N) = -\sqrt{3}$ or $\frac{-1}{\sqrt{3}}$</p> <p>N: $y - 2\sqrt{3} = -\sqrt{3}(x - 4)$</p> <p>N: $y = -\sqrt{3}x + 6\sqrt{3}$ AG</p> <p>or $2\sqrt{3} = -\sqrt{3}(4) + c \Rightarrow c = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$ so N: $y = -\sqrt{3}x + 6\sqrt{3}$</p>	<p>Attempt to differentiate both x and y wrt t to give $\pm p\sin t$ and $\pm q\cos 2t$ respectively</p> <p>Correct $\frac{dx}{dt}$ and $\frac{dy}{dt}$</p> <p>Divides in correct way round and attempts to substitute their value of t (in degrees or radians) into their $\frac{dy}{dx}$ expression.</p> <div style="border: 1px solid black; padding: 5px; margin: 5px 0;"> <p>You may need to check candidate's substitutions for M1*</p> </div> <div style="border: 1px solid black; padding: 5px; margin: 5px 0;"> <p>Note the next two method marks are dependent on M1*</p> </div> <p>Uses $m(N) = -\frac{1}{\text{their } m(T)}$.</p> <p>Uses $y - 2\sqrt{3} = (\text{their } m_N)(x - 4)$ or finds c using $x = 4$ and $y = 2\sqrt{3}$ and uses $y = (\text{their } m_N)x + "c"$.</p> <p>$y = -\sqrt{3}x + 6\sqrt{3}$</p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p> <p>M1*</p> <p>dM1*</p> <p>dM1*</p> <p>A1 cso AG</p>

[2]

[6]

(c) $A = \int_0^4 y \, dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4 \sin 2t \cdot (-8 \sin t) \, dt$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32 \sin 2t \cdot \sin t \, dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32(2 \sin t \cos t) \cdot \sin t \, dt$$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -64 \sin^2 t \cos t \, dt$$

$$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 64 \sin^2 t \cos t \, dt$$

(d) {Using substitution $u = \sin t \Rightarrow \frac{du}{dt} = \cos t$ }

{change limits:

when $t = \frac{\pi}{3}$, $u = \frac{\sqrt{3}}{2}$ & when $t = \frac{\pi}{2}$, $u = 1$ }

$$A = 64 \left[\frac{\sin^3 t}{3} \right]_{\frac{\pi}{2}}^{\frac{\pi}{3}} \quad \text{or} \quad A = 64 \left[\frac{u^3}{3} \right]_{\frac{\sqrt{3}}{2}}^1$$

$$A = 64 \left[\frac{1}{3} - \left(\frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \right) \right]$$

$$A = 64 \left(\frac{1}{3} - \frac{1}{8} \sqrt{3} \right) = \frac{64}{3} - 8\sqrt{3}$$

(Note that $a = \frac{64}{3}$, $b = -8$)

attempt at $A = \int \frac{y}{x} \frac{dx}{dt} \, dt$ M1
 correct expression (ignore limits and dt) A1

Seeing $\sin 2t = 2 \sin t \cos t$ anywhere in PART (c). M1

Correct proof. Appreciation of how the negative sign affects the limits. Note that the answer is given in the question.

A1 AG

[4]

$k \sin^3 t$ or ku^3 with $u = \sin t$ M1
 Correct integration ignoring limits. A1

Substitutes limits of either ($t = \frac{\pi}{2}$ and $t = \frac{\pi}{3}$) or ($u = 1$ and $u = \frac{\sqrt{3}}{2}$) and subtracts the correct way round.

dM1

$$\frac{64}{3} - 8\sqrt{3}$$

A1 aef isw

Aef in the form $a + b\sqrt{3}$, with awrt 21.3 and anything that cancels to $a = \frac{64}{3}$ and $b = -8$.

[4]

16 marks

Q4, (Jan 2010, Q7)

(a)	$y = 0 \Rightarrow t(9 - t^2) = t(3 - t)(3 + t) = 0$			
	$t = 0, 3, -3$		Any one correct value	B1
	At $t = 0$, $x = 5(0)^2 - 4 = -4$		Method for finding one value of x	M1
	At $t = 3$, $x = 5(3)^2 - 4 = 41$			
	(At $t = -3$, $x = 5(-3)^2 - 4 = 41$)			
	At A , $x = -4$; at B , $x = 41$		Both	A1 (3)
(b)	$\frac{dx}{dt} = 10t$		Seen or implied	B1
	$\int y \, dx = \int y \frac{dx}{dt} \, dt = \int t(9 - t^2)10t \, dt$			M1 A1
	$= \int (90t^2 - 10t^4) \, dt$			
	$= \frac{90t^3}{3} - \frac{10t^5}{5} (+C) \quad (= 30t^3 - 2t^5 (+C))$			A1
	$\left[\frac{90t^3}{3} - \frac{10t^5}{5} \right]_0^3 = 30 \times 3^3 - 2 \times 3^5 \quad (= 324)$			M1
	$A = 2 \int y \, dx = 648 \quad (\text{units}^2)$			A1 (6)
				[9]

Q5, (Jan 2013, Q5)

	$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1 \text{ or } y = e^{t \ln 2} - 1$		
(a)	$\{x = 0 \Rightarrow\} 0 = 1 - \frac{1}{2}t \Rightarrow t = 2$ When $t = 2, y = 2^2 - 1 = 3$	Applies $x = 0$ to obtain a value for t .	M1
		Correct value for y .	A1
			[2]
(b)	$\{y = 0 \Rightarrow\} 0 = 2^t - 1 \Rightarrow t = 0$ When $t = 0, x = 1 - \frac{1}{2}(0) = 1$	Applies $y = 0$ to obtain a value for t . (Must be seen in part (b)).	M1
		$x = 1$	A1
			[2]
(c)	$\frac{dx}{dt} = -\frac{1}{2}$ and either $\frac{dy}{dt} = 2^t \ln 2$ or $\frac{dy}{dt} = e^{t \ln 2} \ln 2$ $\frac{dy}{dx} = \frac{2^t \ln 2}{-\frac{1}{2}}$	Attempts their $\frac{dy}{dt}$ divided by their $\frac{dx}{dt}$.	B1 M1
	At A, $t = "2"$, so $m(\mathbf{T}) = -8 \ln 2 \Rightarrow m(\mathbf{N}) = \frac{1}{8 \ln 2}$	Applies $t = "2"$ and $m(\mathbf{N}) = \frac{-1}{m(\mathbf{T})}$	M1
	$y - 3 = \frac{1}{8 \ln 2} (x - 0) \text{ or } y = 3 + \frac{1}{8 \ln 2} x \text{ or equivalent.}$	See notes.	M1 A1 oe cso
			[5]
(d)	$\text{Area}(R) = \int (2^t - 1) \cdot \left(-\frac{1}{2}\right) dt$ $x = -1 \rightarrow t = 4 \text{ and } x = 1 \rightarrow t = 0$	Complete substitution for both y and dx	M1 B1
	$= \left\{-\frac{1}{2}\right\} \left\{\frac{2^t}{\ln 2} - t\right\}$	Either $2^t \rightarrow \frac{2^t}{\ln 2}$ or $(2^t - 1) \rightarrow \frac{(2^t)}{\pm \alpha (\ln 2)} - t$ or $(2^t - 1) \rightarrow \pm \alpha (\ln 2)(2^t) - t$	M1*
	$\left\{-\frac{1}{2} \left[\frac{2^t}{\ln 2} - t \right]_4^0\right\} = -\frac{1}{2} \left(\left(\frac{1}{\ln 2} \right) - \left(\frac{16}{\ln 2} - 4 \right) \right)$	$(2^t - 1) \rightarrow \frac{2^t}{\ln 2} - t$	A1
	$= \frac{15}{2 \ln 2} - 2$	Depends on the previous method mark. Substitutes their changed limits in t and subtracts either way round.	dM1*
		$\frac{15}{2 \ln 2} - 2$ or equivalent.	A1
			[6] 15

Q6, (Edexcel 6666, Sample Paper A3, Q8)

<p>(a) $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4 \cos \theta}{-5 \sin \theta}$</p> <p>Equation of tangent is $y - 4 \sin \alpha = \frac{4 \cos \alpha}{-5 \sin \alpha} (x - 5 \cos \alpha)$</p> <p>$\therefore 5y \sin \alpha + 4x \cos \alpha = 20(\cos^2 \alpha + \sin^2 \alpha) = 20$ (*)</p>	<p>M1 A1</p> <p>M1</p> <p>A1 (4)</p>
<p>(b) $\int y \frac{dx}{d\theta} d\theta = - \int 4 \sin \theta \cdot 5 \sin \theta d\theta$</p> <p style="padding-left: 40px;">$= 10 \int (\cos 2\theta - 1) d\theta$</p> <p style="padding-left: 40px;">$= [5 \sin 2\theta - 10\theta]$</p> <p style="padding-left: 40px;">Area = 20π</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>A1 cso (4)</p>
<p>(c) When $x = 0, y = \frac{4}{\sin \alpha}$, or when $y = 0, x = \frac{5}{\cos \alpha}$</p> <p>Area of parallelogram = $4 \times \frac{10}{\sin \alpha \cos \alpha} = \frac{80}{\sin 2\alpha}$</p> <p>$\therefore A = \frac{80}{\sin 2\alpha} - 20\pi$</p>	<p>B1</p> <p>M1 A1</p> <p>A1 (4)</p>
<p>(d) $\frac{80}{\sin 2\alpha} - 20\pi = 20\pi$</p> <p>$\sin 2\alpha = \frac{2}{\pi}$</p> <p>$\alpha = 0.345$</p>	<p>M1 A1</p> <p>A1 (3)</p>
<p>(15 marks)</p>	

Q7, (Edexcel 6666, Sample Paper A6, Q4)

<p>(a) Area of triangle = $\frac{1}{2} \times 30 \times 3\pi^2$ (= 444.132)</p>	<p>Accept 440 or 450</p>	<p>B1 (1)</p>
<p>(b) Area shaded = $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 30 \sin 2t \cdot 32t dt$</p> <p style="padding-left: 40px;">$= [-480t \cos 2t + \int 480 \cos 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$</p> <p style="padding-left: 80px;">$= [-480t \cos 2t + 240 \sin 2t]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$</p> <p style="padding-left: 40px;">$= 240(\pi - 1)$</p>	<p>M1 A1</p> <p>M1 A1</p> <p>A1 ft</p> <p>M1A1 (7)</p>	<p>(1)</p>
<p>(c) Percentage error = $\frac{240(\pi - 1) - estimate}{240(\pi - 1)} \times 100 = 13.6\%$ (Accept answers in the range 12.4% to 14.4%)</p>	<p>M1 A1 (2)</p>	<p>(2)</p>
<p>(10 marks)</p>		