The Exponential Distribution (From AQA MS04)

Q1, (Jun 2008, Q4)

The lifetimes of electrical components follow an exponential distribution with mean 200 hours.

(a) Calculate the probability that the lifetime of a randomly selected component is:

(i) less than 120 hours; *(2 marks)*

(ii) more than 160 hours; *(2 marks)*

(iii) less than 160 hours, given that it has lasted more than 120 hours. *(3 marks)*

(b) Determine the median lifetime of these electrical components. *(3 marks)*

Q2, (Jun 2009, Q7)

The continuous random variable X is modelled by an exponential distribution with probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

(a) Write down the cumulative distribution function, $F(x)$. *(2 marks)*

(b) Show that the **exact** value of the interquartile range of X is given by $\frac{1}{\lambda}\ln 3$. *(5 marks)*

(c) (i) Use integration to prove that $E(X^2) = \frac{2}{\lambda^2}$. *(4 marks)*

(ii) Hence, given that $E(X) = \frac{1}{\lambda}$, show that $\text{Var}(X) = \frac{1}{\lambda^2}$. *(1 mark)*

(d) (i) Find the **exact** value of λ for which the value of the interquartile range of X is four times the value of the variance of X. *(2 marks)*

(ii) Describe what happens to the interquartile range of X as λ increases without bound. *(1 mark)*

Q3, (Jun 2011, Q6)

(a) The continuous random variable X follows an exponential distribution if it has a probability density function

$$f(x) = \begin{cases} ke^{-kx} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive constant.

(i) Prove that the mean value, μ, of X is $\frac{1}{k}$. *(3 marks)*

(ii) Find, in terms of k, the median value, m, of X and hence show that $m < \mu$. *(5 marks)*

(b) The number of radioactive particles striking a screen in a time period of length t seconds follows a Poisson distribution with mean $\frac{t}{\lambda}$, where λ is a constant.

(i) Write down the probability that no particles strike the screen in a period of t seconds. *(1 mark)*

(ii) The random variable T is defined as the length of time, in seconds, between successive radioactive particles striking the screen.

(A) Show that

$$P(T < t) = 1 - e^{-\frac{t}{\lambda}}$$ *(2 marks)*

(B) Hence, by finding the probability density function of T, state the distribution of T. *(2 marks)*

Q4, (Jun 2013, Q4)

The random variable X has an exponential distribution with mean μ. The cumulative distribution function of X for $x \geqslant 0$ is given by

$$F(x) = 1 - e^{-\frac{1}{\mu}x}$$

(a) Find an exact expression for the interquartile range of X in terms of μ. *(5 marks)*

(b) Prove, by integration, that $E(X^2) = 2\mu^2$. *(4 marks)*

(c) Show that the standard deviation of X is less than the interquartile range of X. *(3 marks)*

Q5, (Jun 2012, Q5)

A random variable X has an exponential distribution with probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} k\mathrm{e}^{-kx} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and k is a constant.

(a) Given that $\mathrm{E}(X) = \dfrac{1}{k}$, find:

(i) using integration, $\mathrm{E}(X^2)$;

(ii) $\mathrm{Var}(X)$. *(6 marks)*

(b) (i) Derive the cumulative distribution function, $\mathrm{F}(x)$, of X for $x \geqslant 0$. *(3 marks)*

(ii) Hence find, in terms of k, the **exact** value of the 90th percentile of X. *(3 marks)*

(c) A machine has two essential components, the lifetimes of which follow exponential distributions with means a hours and $3a$ hours. The machine will stop if either component fails. The failures of the two components may be taken to be independent.

Find the probability that the machine continues to work for at least a hours from the start, giving your answer in the form e^q, where q is a rational number to be determined. *(4 marks)*

Q6, (Jun 2014, Q1)

The continuous random variable T has probability density function $\mathrm{f}(t)$, where

$$\mathrm{f}(t) = \begin{cases} 5\mathrm{e}^{-5t} & t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Derive the cumulative distribution function of T.

[4 marks]

(b) Find the probability that $T > \mathrm{E}(T)$.

[1 mark]

(c) Find the value of the constant c such that $\mathrm{P}(T > c) = 0.05$.

[2 marks]

Q7, (Jun 2015, Q4)

(a) The random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Use integration to find an expression for:

(A) $E(X)$;

(B) $P(X > x)$ for $x \geqslant 0$.

[5 marks]

(ii) Given that m denotes the median of X, evaluate $P(m < X \leqslant \mu)$.

[2 marks]

(b) During the manufacture of barbed wire, the length, X kilometres, between successive faults may be modelled by an exponential distribution with mean 2.

(i) Determine the probability that the length between successive faults is between 250 metres and 1250 metres.

[3 marks]

(ii) A farmer purchases 6 reels, each containing 250 metres of barbed wire.

Calculate the probability that at least 5 of the 6 reels contain wire with no faults.

[4 marks]

Q8, (Jun 2016, Q1)

The random variable X has an exponential distribution with mean 16.

Find:

(a) $P(X < 10)$;

(b) $P(10 < X < 20)$;

(c) $P(X \neq 15)$.

[5 marks]

Q9, (Jun 2017, Q5)

(a) The continuous random variable X has the cumulative distribution function $F(x)$ where

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & 0 \leqslant x < \infty \end{cases}$$

and $E(X) = \frac{1}{\lambda}$.

(i) Deduce the probability density function, $f(x)$, of X for $0 \leqslant x < \infty$.

[1 mark]

(ii) Use integration to find an expression for $E(X^2)$.

[3 marks]

(iii) Hence show that $\text{Var}(X) = \frac{1}{\lambda^2}$.

[1 mark]

(b) The number of emails received by a helpdesk may be modelled by a Poisson distribution with an average of 3 emails per hour.

(i) Determine the probability that the helpdesk receives fewer than 15 emails during a four-hour period.

[2 marks]

(ii) Calculate the probability that the time between successive emails is:

(A) exactly 20 minutes;

(B) less than 15 minutes;

(C) between 15 and 25 minutes.

[6 marks]