**Continuous Random Variables (Cumulative Distributions) (From OCR 4768)**

**Q1, (Jan 2006, Q1i-iii)**

A railway company is investigating operations at a junction where delays often occur. Delays (in minutes) are modelled by the random variable $T$ with the following cumulative distribution function.

$$F(t)=\begin{cases}0 & t\leqslant 0\\ 1-e^{-\frac{1}{3}t} & t>0\end{cases}$$

**(i)** Find the median delay and the 90th percentile delay. [5]

**(ii)** Derive the probability density function of $T$. Hence use calculus to find the mean delay. [5]

**(iii)** Find the probability that a delay lasts longer than the mean delay. [2]

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**Q2, (Jun 2006, Q1i,ii)**

Design engineers are simulating the load on a particular part of a complex structure. They intend that the simulated load, measured in a convenient unit, should be given by the random variable $X$ having probability density function

$$f(x) = 12x^3 - 24x^2 + 12x, \qquad 0 \leqslant x \leqslant 1.$$

**(i)** Find the mean and the mode of $X$. [6]

**(ii)** Find the cumulative distribution function $F(x)$ of $X$.

Verify that $F\left(\frac{1}{4}\right)=\frac{67}{256}$, $F\left(\frac{1}{2}\right)=\frac{11}{16}$ and $F\left(\frac{3}{4}\right)=\frac{243}{256}$. [3]

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**Q3, (Jan 207, Q1i-iv)**

The continuous random variable $X$ has probability density function

$$f(x) = k(1-x) \quad \text{for } 0 \leqslant x \leqslant 1$$

where $k$ is a constant.

**(i)** Show that $k = 2$. Sketch the graph of the probability density function. [4]

**(ii)** Find $E(X)$ and show that $\text{Var}(X) = \frac{1}{18}$. [5]

**(iii)** Derive the cumulative distribution function of $X$. Hence find the probability that $X$ is greater than the mean. [4]

**(iv)** Verify that the median of $X$ is $1-\dfrac{1}{\sqrt{2}}$. [2]

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**Q4, (Jun 2008, Q1a)**

Sarah travels home from work each evening by bus; there is a bus every 20 minutes. The time at which Sarah arrives at the bus stop varies randomly in such a way that the probability density function of $X$, the length of time in minutes she has to wait for the next bus, is given by

$$f(x) = k(20 - x) \text{ for } 0 \leqslant x \leqslant 20, \text{ where } k \text{ is a constant.}$$

**(i)** Find $k$. Sketch the graph of $f(x)$ and use its shape to explain what can be deduced about how long Sarah has to wait. **[5]**

**(ii)** Find the cumulative distribution function of $X$ and hence, or otherwise, find the probability that Sarah has to wait more than 10 minutes for the bus. **[4]**

**(iii)** Find the median length of time that Sarah has to wait. **[3]**

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**Q5, (Jun 2009, Q4i-iii)**

A random variable $X$ has probability density function $f(x) = \frac{2x}{\lambda^2}$ for $0 < x < \lambda$, where $\lambda$ is a positive constant.

**(i)** Show that, for any value of $\lambda$, $f(x)$ is a valid probability density function. **[3]**

**(ii)** Find $\mu$, the mean value of $X$, in terms of $\lambda$ and show that $P(X < \mu)$ does not depend on $\lambda$. **[4]**

**(iii)** Given that $E(X^2) = \frac{\lambda^2}{2}$, find $\sigma^2$, the variance of $X$, in terms of $\lambda$. **[2]**

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**Q6, (Jun 2010, Q4i,ii)**

A random variable $X$ has an exponential distribution with probability density function $f(x) = \lambda e^{-\lambda x}$ for $x \geqslant 0$, where $\lambda$ is a positive constant.

**(i)** Verify that $\int_0^\infty f(x)\,dx = 1$ and sketch $f(x)$. **[5]**

**(ii)** In this part of the question you may use the following result.

$$\int_0^\infty x^r e^{-\lambda x}\,dx = \frac{r!}{\lambda^{r+1}} \quad \text{for } r = 0, 1, 2, \ldots .$$

Derive the mean and variance of $X$ in terms of $\lambda$. **[6]**

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**Q7, (Jan 2013, Q2i,ii)**

A particular species of reed that grows up to 2 metres in length is used for thatching. The lengths in metres of the reeds when harvested are modelled by the random variable $X$ which has the following probability density function, $f(x)$.

$$f(x) = \begin{cases} \frac{3}{16}(4x - x^2) & \text{for } 0 \leqslant x \leqslant 2 \\ 0 & \text{elsewhere} \end{cases}$$

**(i)** Sketch $f(x)$. **[3]**

**(ii)** Show that $E(X) = \frac{5}{4}$ and find the standard deviation of the lengths of the harvested reeds. **[8]**

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**Q8, (Jun 2011, Q3)**

The time, in hours, until an electronic component fails is represented by the random variable $X$. In this question two models for $X$ are proposed.

**(i)** In one model, $X$ has cumulative distribution function

$$G(x) = \begin{cases} 0 & x \leqslant 0, \\ 1 - \left(1 + \frac{x}{200}\right)^{-2} & x > 0. \end{cases}$$

($A$) Sketch $G(x)$. **[3]**

($B$) Find the interquartile range for this model. Hence show that a lifetime of more than 454 hours (to the nearest hour) would be classed as an outlier. **[6]**

**(ii)** In the alternative model, $X$ has probability density function

$$f(x) = \begin{cases} \frac{1}{200}e^{-\frac{1}{200}x} & x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

($A$) For this model show that the cumulative distribution function of $X$ is

$$F(x) = \begin{cases} 0 & x \leqslant 0, \\ 1 - e^{-\frac{1}{200}x} & x > 0. \end{cases}$$ **[3]**

($B$) Show that $P(X > 50) = e^{-0.25}$. **[2]**

($C$) It is observed that a particular component is still working after 400 hours. Find the conditional probability that it will still be working after a further 50 hours (i.e. after a total of 450 hours) given that it is still working after 400 hours. **[4]**

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**Q9, (Jun 2012, Q4ii-v)**

The time $T$, measured in days, that elapses between successive call-outs can be modelled using the exponential distribution for which $f(t)$, the probability density function, is

$$f(t) = \begin{cases} 0 & t < 0, \\ \lambda e^{-\lambda t} & t \geqslant 0, \end{cases}$$

where $\lambda$ is a positive constant.

**(ii)** For the distribution above, it can be shown that $E(T) = \frac{1}{\lambda}$. Given that the mean time between successive call-outs is $\frac{5}{3}$ days, write down the value of $\lambda$. **[1]**

**(iii)** Find $F(t)$, the cumulative distribution function. **[3]**

**(iv)** Find the probability that the time between successive call-outs is more than 1 day. **[2]**

**(v)** Find the median time that elapses between successive call-outs. **[3]**

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**Q10, (Jun 2014, Q4i-iii)**

The probability density function of a random variable $X$ is given by

$$f(x) = \begin{cases} kx & 0 \leqslant x \leqslant a, \\ k(2a-x) & a < x \leqslant 2a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a$ and $k$ are positive constants.

**(i)** Sketch f($x$). Hence explain why E($X$) = $a$. **[3]**

**(ii)** Show that $k = \frac{1}{a^2}$. **[3]**

**(iii)** Find Var($X$) in terms of $a$. **[4]**

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**Q11, (Jun 2016, Q3i-iv)**

The random variable $X$ has the following probability density function:

$$f(x) = \begin{cases} k(1-x^2) & -1 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere,} \end{cases}$$

where $k$ is a positive constant.

**(i)** Calculate the value of $k$. **[3]**

**(ii)** Sketch the probability density function. **[3]**

**(iii)** Calculate Var($X$). **[3]**

**(iv)** Find a cubic equation satisfied by the upper quartile $q$, and hence verify that $q = 0.35$ to 2 decimal places. **[5]**

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