Goodness of Fit Tests (Year 2) (From OCR 4768)

Q1, (Jun 2006, Q1)

Q1	$f(x) = 12x^3 - 24x^2 + 12x, \quad 0 \le x \le 1$			
(i)	$E(X) = \int_0^1 x f(x)dx$	M1	Integral for E(*X*) including limits (which may appear later).	
	$= 12\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$	A1	Successfully integrated.	
	$= 12\left[\frac{1}{5} - \frac{2}{4} + \frac{1}{3}\right] = 12 \times \frac{1}{30} = \frac{2}{5}$	A1	Correct use of limits leading to final answer. C.a.o.	
	For mode, $f'(x) = 0$	M1		
	$f'(x) = 12(3x^2 - 4x + 1) = 12(3x-1)(x-1)$ $\therefore f'(x) = 0$ for $x = 1$ and $x = \frac{1}{3}$	A1		
	Any convincing argument (e.g. $f''(x)$) that $\frac{1}{3}$ (and not 1) is the mode.	A1		6
(ii)	Cdf $F(x) = \int_0^x f(t)dt$ $= 12\left(\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2}\right)$	M1	Definition of cdf, including limits (or use of "+c" and attempt to evaluate it), possibly implied later. Some valid method must be seen.	
	$= 3x^4 - 8x^3 + 6x^2$	A1	Or equivalent expression; condone absence of domain [0,1].	
	$F\left(\frac{1}{4}\right) = \frac{3}{256} - \frac{8}{64} + \frac{6}{16} = \frac{3-32+96}{256} = \frac{67}{256}$ $F\left(\frac{1}{2}\right) = \frac{3}{16} - \frac{8}{8} + \frac{6}{4} = \frac{3-16+24}{16} = \frac{11}{16}$ $F\left(\frac{3}{4}\right) = \frac{3 \times 81}{256} - \frac{8 \times 27}{64} + \frac{6 \times 9}{16} = \frac{243}{256}$	B1	For all three; answers given; must show convincing working (such as common denominator)! Use of decimals is not acceptable.	3
(iii)	(table below)	B2	For e_i. B1 if any 2 correct, provided Σ = 512.	
	$X^2 = 0{\cdot}4776 + 0{\cdot}3716 + 0{\cdot}0672 + 15{\cdot}3846 = 16{\cdot}30(1)$	M1 A1		
	Refer to χ_3^2.	M1	Must be some clear evidence of reference to χ_3^2, probably implicit by reference to a critical point (5% : 7·815; 1% : 11·34). No ft (to the A marks) if incorrect χ^2 used, but E marks are still available.	
	Very highly significant. Very strong evidence that the model does not fit.	A1 A1	There must be at least one reference to "very …", i.e. the extremeness of the test statistic.	
	The main feature is that we observe many more loads at the "top end" than expected.	E1	Or e.g. "big/small" contributions to X^2 gets E1, …	
	The other observations are below expectation, but discrepancies are comparatively small.	E1	… and directions of discrepancies gets E1.	9
				18

o_i	126	209	131	46
e_i	134	352 – 134 = 218	486 – 352 = 134	26

Q2, (Jan 2007, Q4a)

Obs	Exp
10	6·68

$\therefore X^2 = \frac{(10-6\cdot68)^2}{6\cdot68} + \text{etc}$	M1	Combine first two rows.	
= 1·6501 + 1·7740 + 3·3203 + 4·5018 + 0·4015 + 0·8135	M1		
= 12·46(12)	A1		
d.o.f. = 6 – 3 = 3		Require d.o.f. = No. cells used – 3.	
Refer to χ_3^2.	M1	No ft from here if wrong.	
Upper 5% point is 7·815	A1	No ft from here if wrong.	
12·46 > 7·815 ∴ Result is significant.	E1	ft only c's test statistic.	
Seems the Normal model does not fit the data at the 5% level.	E1	ft only c's test statistic.	
E.g. • The biggest discrepancy is in the class $1\cdot01 < a \le 1\cdot02$	E1		
• The model overestimates in classes …, but underestimates in classes …	E1	Any two suitable comments.	9

Q3, (Jun 2007, Q4i)

Obs	21	24	12	15	13	9	6
Exp	26·53	17·22	20·25	11·00	10·94	8·74	5·32

	M1	Probabilities × 100.	
	A1	All Expected frequencies correct.	
$\therefore X^2 = \frac{(21-26\cdot53)^2}{26\cdot53} + \text{etc}$	M1		
= 1·1527 + 2·6695 + 3·3611 + 1·4545 + 0·3879 + 0·0077+ 0·0869	A1	At least 4 values correct.	
= 9·1203	A1		
d.o.f. = 7 – 1 = 6			
Refer to χ_6^2.	M1	No ft from here if wrong.	
Upper 5% point is 12·59	A1	No ft from here if wrong.	
9·1203 < 12·59 ∴ Result is not significant.	E1	ft only c's test statistic.	
Evidence suggests the model fits the data at the 5% level.	E1	ft only c's test statistic.	9

Q4, (Jan 2008, Q4a)

(a) (i)	$\bar{x} = \frac{1125}{500} = 2.25$	B1		
	For binomial $E(X) = n \times p$	M1	Use of mean of binomial distribution. May be implicit.	
	$\therefore \hat{p} = \frac{2.25}{5} = 0.45$	A1	Beware: answer given.	3
(ii)				

f_o	32	110	154	125	63	16
f_e (calc)	25.164	102.944	168.455	137.827	56.384	9.226
f_e (tables)	25.15	102.95	168.45	137.85	56.35	9.25

	M1	Calculation of expected frequencies.	
	A1	All correct.	
X^2 = 1.8571 + 0.4836 + 1.2404 + 1.1938 + 0.7763 + 4.9737	M1	Or using tables: 1.8657 + 0.4828 + 1.2396 + 1.1978 + 0.7848 + 4.9257	
= 10.52(49)	A1	c.a.o. Or using tables: 10.49(64)	
Refer to χ_4^2.	M1	Allow correct df (= cells – 2) from wrongly grouped or ungrouped table, and FT. Otherwise, no FT if wrong.	
Upper 5% point is 9.488.	A1	No ft from here if wrong.	
Significant.	A1	ft only c's test statistic.	
Suggests binomial model does not fit.	A1	ft only c's test statistic.	
The model appears to overestimate in the middle and to underestimate at the tails.	E1	Accept also any other sensible comment e.g. at 2.5% significance, the result would NOT have been significant.	
The biggest discrepancy is at $X = 5$.	E1		
A binomial model assumes all trials are independent with a constant probability of "success". It seems unlikely that there will be independence within families and/or that p will be the same for all families.	E2	(E2, 1, 0) Any sensible comment which addresses independence and constant p.	12

Q5, (Jun 2008, Q4a)

(a) (i)	$\bar{x} = \frac{310}{100} = 3.1$	B1		
	$s^2 = \frac{1288 - 100 \times 3.1^2}{99} = \frac{327}{99} = 3.303$	B1		
	Evidence could support Poisson since the variance is fairly close to the mean.	E1		3

(ii)

f_o	6	16	19	18	17	14	6	4	0
f_e	4.50	13.97	21.65	22.37	17.33	10.75	5.55	2.46	1.42
Merged	22 18.47						10 9.43		

	M1 A1 A1	Calculation of expected frequencies. Last cell correct. All others correct, but ft if wrong.	
	M1	Combining cells. (Condone if not combined as fully as shown above, but require top two cells combined as a minimum.)	
X^2 = 0.6747 + 0.3244 + 0.8537 + 0.0063 + 0.9826 + 0.0345	M1	Calculation of X^2.	
= 2.876(2)	A1	(Condone wrong last cell.) Depends on both of the preceding M marks.	
Refer to χ_4^2. e.g. Upper 10% point is 7.779.	M1	Allow correct df (= cells – 2) from wrongly grouped or ungrouped table, and FT. Otherwise, no FT if wrong.	
Not significant.	A1	ft only c's test statistic.	
Suggests Poisson model does fit …	A1	ft only c's test statistic.	
… at any reasonable level of significance.	A1	Or other sensible comment.	10

Q6, (Jan 2010, Q1)

(i)	H_0: The number of eggs hatched can be modelled by B(3, ½)	B1		
	H_1: The number of eggs hatched cannot be modelled by B(3, ½)	B1		

With $p = ½$

Probability	0.125	0.375	0.375	0.125
Exp'd frequency	10	30	30	10
Obs'd frequency	7	23	29	21

		M1	Probs × 80 for expected frequencies.	
		A1	All correct.	
	$X^2 = 0.9 + 1.6333 + 0.0333 + 12.1$	M1	Calculation of X^2.	
	$= 14.666(7)$	A1	c.a.o.	
	Refer to χ_3^2.	M1	Allow correct df (= cells – 1) from wrongly grouped table and ft. Otherwise, no ft if wrong. $P(X^2 > 14.667) = 0.00212$.	
	Upper 5% point is 7.815.	A1	No ft from here if wrong.	
	Significant.	A1	ft only c's test statistic.	
	Suggests it is reasonable to suppose model with $p = ½$ does not apply.	A1	ft only c's test statistic.	[10]
(ii)	$\bar{x} = \frac{144}{80} = 1.8$	B1	C.a.o.	
	$\therefore \hat{p} = \frac{1.8}{3} = 0.6$	B1	Use of $E(X) = np$. ft c's mean, provided $0 < \hat{p} < 1$.	[2]
(iii)	Refer to χ_2^2.	M1	Allow df 1 less than in part (i). No ft if wrong.	
	Upper 5% point is 5.991.	A1	No ft if wrong.	
	Suggests it is reasonable to suppose model with estimated p does apply.	A1	ft provided previous A mark awarded.	[3]
(iv)	For example: Estimating p leads to an improved fit … … at the expense of the loss of 1 degree of freedom. The model in (i) fails due to a large underestimate for $X = 3$.	E2	Reward any two sensible points for E1 each.	[2]
			Total	**[17]**

Q7, (Jun 2014, Q3b)

Answer	Marks	Guidance
H_0 : The Poisson model fits the data H_1 : The Poisson model does not fit the data	B1	Both hypotheses. Must be the right way round. Do not accept "data fits model" or equivalent.
	M1	At least 3 probabilities to 3dp or better or 3 expected values to 3sf or better
	M1	Multiply by 60 to obtain expected values
	A1	All correct to 3sf or better
	M1	Merge first 2 and last 2 cells

r	$P(X = r)$	Expected value	Combined
0	0.03688	2.213	9.516
1	0.12171	7.303	
2	0.20083	12.050	
3	0.22091	13.255	
4	0.18225	10.935	
5	0.12029	7.217	
6	0.06616	3.969	7.027
≥ 7	0.05097	3.058	

$$X^2 = \frac{2.516^2}{9.516} + \frac{2.050^2}{12.050} + \frac{3.745^2}{13.255} + \frac{3.065^2}{10.935} + \frac{0.217^2}{7.217} + \frac{2.027^2}{7.027}$$

$= 0.6652 + 0.3488 + 1.0581 + 0.8591 + 0.0065 + 0.5847$ — M1* — Calculation of X^2

$= 3.522$ awrt 3.52 — A1 — cao

Refer to χ_4^2 — M1 — Allow correct df from wrongly grouped table.

Upper 5% point is 9.49 — B1 — No FT from here if wrong.

$3.522 < 9.49$ cannot reject H_0 — A1dep — FT candidates 3.522 if relevant M1 earned.

Poisson model appears to fit data. — A1dep — FT candidates 3.522 if relevant M1 earned. Do not accept "data fits Poisson model" or equivalent.

[11]

Q8, (Jan 2011, Q3)

(i)	Using mid- intervals 1.5, 1.7, etc $\bar{x} = \frac{205}{100} = 2.05$ $s = \sqrt{\frac{425.16 - 100 \times 2.05^2}{99}} = 0.2227(01...)$	M1 A1 E1	 Mean. s.d. Answer given; must show convincingly.	3
(ii)	$f = 100 \times P(1.8 \le M < 2.0)$ $= 100 \times P(-1.1226 \le z < -0.2245)$ $= 100 \times ((1 - 0.5888) - (1 - 0.8691))$ $= 100 \times (0.4112 - 0.1309) = 28.03$	M1 A1 A1	Probability × 100. Correct Normal probabilities. ft c's mean. Must show convincingly using Normal distribution. ft c's mean.	3
(iii)	H_0: The Normal model fits the data. H_1: The Normal model does not fit the data. $X^2 = 0.7294 + 0.1384 + 1.9623 + 3.5155 + 0.2437$ $= 6.589(3)$ Refer to χ_2^2. Upper 5% point is 5.991. Significant. Evidence suggests that the model does not fit the data.	B1 B1 M1 M1 A1 M1 A1 A1 A1	Ignore any reference to parameters. Merge first 2 and last 2 cells. Calculation of X^2. c.a.o. Allow correct df (= cells – 3) from wrongly grouped table and ft. Otherwise, no ft if wrong. $P(X^2 > 6.589) = 0.0371$. No ft from here if wrong. ft only c's test statistic. ft only c's test statistic. Conclusion in context.	9
(iv)	The model • overestimates in the 2.2 – 2.4 class, • underestimates in the 2 – 2.2 class. At lower significance levels the test would not have been significant.	 E1 E1 E1		3
				18

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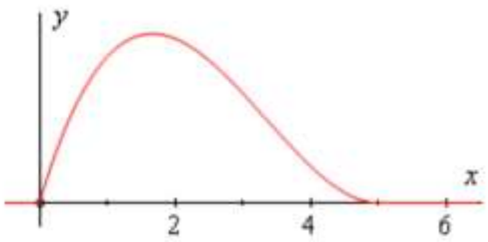
Q9, (Jun 2013, Q3)

(i)		G1 G1 G1 **[3]**	Curve, through the origin and in the first quadrant only. A single maximum; curve returns to $y = 0$; nothing to the right of $x = 5$. No t.pt at $x = 0$; t.pt. at $x = 5$; (5, 0) labelled (p.i. by an indicated scale).
(ii)	$\mathrm{F}(x) = k\int_0^x t(t-5)^2\,\mathrm{d}t$ $= k\left[\frac{t^4}{4} - \frac{10t^3}{3} + \frac{25t^2}{2}\right]_0^x$ $= k\left(\frac{x^4}{4} - \frac{10x^3}{3} + \frac{25x^2}{2}\right)$	M1 M1 A1 **[3]**	Correct integral for F(x) with limits (which may appear later). Correctly integrated. Limits used correctly to obtain expression. Condone absence of "–0". Do not require complete definition of F(x). Dependent on both M1's
(iii)	$\mathrm{F}(5) = 1$ $\therefore k\left(\frac{5^4}{4} - \frac{10\times 5^3}{3} + \frac{25\times 5^2}{2}\right) = 1$ $\therefore k\left(\frac{1875 - 5000 + 3750}{12}\right) = 1$ $\therefore k \times \frac{625}{12} = 1$ $\therefore k = \frac{12}{625}$	M1 A1 **[2]**	Substitute $x = 5$ and equate to 1. Expect to see evidence of at least this line of working (oe) for A1. Convincingly shown. Beware printed answer.
(iv)	For $0 \le x < 1$, Expected f = 60 × F(1) $= 60 \times \frac{12}{625}\left(\frac{1^4}{4} - \frac{10\times 1^3}{3} + \frac{25\times 1^2}{2}\right) = 10.848$ For $1 \le x < 2$, Expected f = 60 – Σ(the rest) = 20.64	M1 A1 B1 **[3]**	Use of 60 × F(x) with correct k. Allow also 31.488 – frequency for $1 \le x < 2$ provided that one found using F(x). Allow either frequency found by integration. FT 31.488 – previous answer. Or allow 60 × (F(2) – F(1))

(v)	H_0: The model is suitable / fits the data. H_1: The model is not suitable / does not fit the data.	B1	Both hypotheses. Must be the right way round. Do not accept "data fit model" oe.
	Merge last 2 cells: Obs f = 17, Exp f = 10.752	M1	
	$X^2 = 3.1525 + 1.5411 + 1.5460 + 3.6307$	M1	Calculation of X^2.
	$= 9.870$	A1	c.a.o.
	Refer to χ_3^2.	M1	Allow correct df (= cells – 1) from wrongly grouped table and ft. Otherwise, no ft if wrong.
	Upper 2.5% point is 9.348.	A1	No ft from here if wrong. $P(X^2 > 9.870) = 0.0197$.
	Significant.	A1	ft only c's test statistic.
	Sufficient evidence to suggest that the model is not suitable in this context.	A1	ft only c's test statistic. Conclusion in context. Do not accept "data do not fit model" oe.
		[8]	