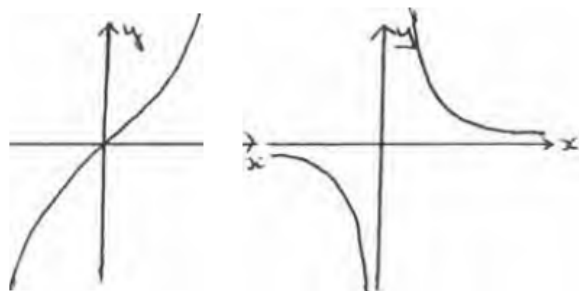


Integration Involving Trigonometric and Hyperbolic Functions (From OCR 4726)

**Q1, (Jan 2007, Q4)**

(i)



B1 Correct shape for  $\sinh x$

B1 Correct shape for  $\operatorname{cosech} x$

B1 Obvious point ( $dy/dx \neq 0$ )/asymptotes clear

(ii) Correct definition of  $\sinh x$

Invert and mult. by  $e^x$  to AG.

Sub.  $u = e^x$  and  $du = e^x dx$

Replace to  $2/(u^2 - 1) du$

Integrate to  $\ln((u - 1)/(u + 1))$

Replace  $u$

B1 May be implied

B1 Must be clear; allow  $2/(e^x \cdot e^{-x})$  as minimum simplification

M1 Or equivalent, all  $x$  eliminated and not  $dx = du$

A1

A1√ Use formulae book, PT, or  $\operatorname{atanh}^{-1}u$

A1 No need for  $c$

**Q2, (Jan 2008, Q9)**

(i) Get  $\sinh y \frac{dy}{dx} = 1$

Replace  $\sinh y = \sqrt{\cosh^2 y - 1}$

Justify positive grad. to A.G.

M1 Or equivalent; allow  $\pm$

Allow use of  $\ln$  equivalent with Chain Rule

A1

B1 e.g. sketch

(ii) Get  $k \cosh^{-1} 2x$

Get  $k = 1/2$

M1 No need for  $c$

A1

(iii) Sub.  $x = k \cosh u$

Replace all  $x$  to  $\int k_1 \sinh^2 u du$

Replace as  $\int k_2 (\cosh 2u - 1) du$

Integrate correctly

Attempt to replace  $u$  with  $x$  equivalent

Tidy to reasonable form

M1

A1

M1 Or exponential equivalent

A1√ No need for  $c$

M1 In their answer

A1 cao  $(\frac{1}{2}x\sqrt{4x^2 - 1}) - \frac{1}{4} \cosh^{-1} 2x (+c)$

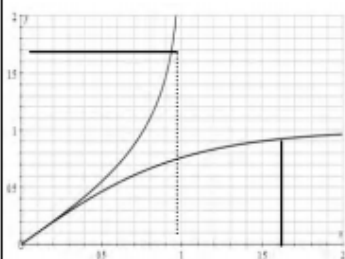
**Q3, (Jan 2009, Q4)**

<p>(i) Let <math>x = \cosh \theta</math> such that  <math>dx = \sinh \theta d\theta</math>                  Clearly use <math>\cosh^2 - \sinh^2 = 1</math></p>	<p>M1 A1</p>	<p>Clearly derive A.G.</p>	
<p>(ii) Replace <math>\cosh^2 \theta</math>                  Attempt to integrate their expression                  Get <math>\frac{1}{4} \sinh 2\theta + \frac{1}{2} \theta (+c)</math>                  Clearly replace for <math>x</math> to A.G.</p>	<p>M1 M1 A1 B1</p>	<p>Allow <math>a (\cosh 2\theta \pm 1)</math>                  Allow <math>b \sinh 2\theta \pm a\theta</math>                  Condone no <math>+c</math>                  SC Use expo. def<sup>n</sup>; three terms                  Attempt to integrate                  Get <math>\frac{1}{8}(e^{2\theta} - e^{-2\theta}) + \frac{1}{2}\theta (+c)</math>                  Clearly replace for <math>x</math> to A.G.</p>	<p>M1 M1 A1 B1</p>

**Q4, (Jun 2009, Q6)**

<p>Get <math>k \sinh^{-1} k_1 x</math></p> <p>Get <math>\frac{1}{3} \sinh^{-1} \frac{3}{4} x</math>                  Get <math>\frac{1}{2} \sinh^{-1} \frac{2}{3} x</math>                  Use limits in their answers                  Attempt to use correct ln laws to set up a solvable equation in <math>a</math>                  Get <math>a = 2^{1/5} \cdot 3^{1/2}</math></p>	<p>M1 A1 A1 M1 M1 A1</p>	<p>For either integral; allow attempt at ln version here                  Or ln version                  Or ln version                  Or equivalent</p>	
---	--	---	--

(iii)



Areas shown are equal:  
 $x = \tanh k$   
 $\Rightarrow y = k$

$$\Rightarrow \int_0^{\tanh k} \tanh^{-1} x \, dx$$

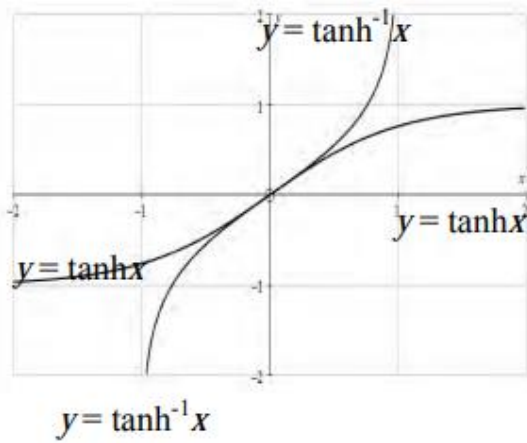
$$= \text{rectangle } (k \times \tanh k) - \text{(ii)}$$

$$= k \tanh k - \ln(\cosh k)$$

<p>M1 A1</p>	<p>M1 A1</p>	<p>For consideration of areas                  For sufficient justification</p>
<p>M1 A1</p>	<p>M1 A1</p>	<p>For subtraction from rectangle                  For correct answer <b>N.B. answer given</b></p>
<p>4</p>	<p>Alternative: Otherwise by parts,                  as <math>1 \times \tanh^{-1} x</math> OR <math>1 \times \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)</math></p>	

**Q5, (Jun 2011, Q7)**

**7(i)**



**B1**

Both curves of the correct shape (ignore overlaps) and labelled

**B1**

gradient = 1 at  $x = 0$  stated

**B1**

For asymptotes  $y = \pm 1$  and  $x = \pm 1$  (or on sketch)

**B1**

Sketch all correct

**4**

**(ii)**

$$\int_0^k \tanh x \, dx = [\ln(\cosh x)]_0^k = \ln(\cosh k)$$

**M1**

For substituting limits into  $\ln \cosh x$

**A1**

For correct answer

**2**

<p><b>7(iii)</b> Alternative method 1 By parts:</p> $I = \int_0^{\tanh k} \tanh^{-1} x \, dx$ $u = \tanh^{-1} x \quad dv = dx$ $du = \frac{1}{1-x^2} dx \quad v = x$ $\Rightarrow I = \left[ x \tanh^{-1} x \right]_0^{\tanh k} - \int_0^{\tanh k} \frac{x}{1-x^2} dx$ $= k \tanh k + \frac{1}{2} \left[ \ln(1-x^2) \right]_0^{\tanh k}$ $= k \tanh k + \frac{1}{2} \ln(1 - \tanh^2 k)$ $= k \tanh k + \frac{1}{2} \ln(\operatorname{sech}^2 k)$ $= k \tanh k + \ln(\operatorname{sech} k)$	<p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p>	<p>For integrating by parts (correct way round)</p> <p>For getting this far</p> <p>Dealing with the resulting integral</p>
<p>Alternative method 2 By substitution Let <math>y = \tanh^{-1} x \Rightarrow x = \tanh y</math> <math>\Rightarrow dx = \operatorname{sech}^2 y \, dy</math> When <math>x = 0, y = 0</math> When <math>x = \tanh k, y = k</math></p> $\Rightarrow I = \int_0^{\tanh k} \tanh^{-1} x \, dx = \int_0^k y \operatorname{sech}^2 y \, dy$ $u = y \quad dv = \operatorname{sech}^2 y \, dy$ $du = dy \quad v = \tanh y$ $\Rightarrow I = \left[ y \tanh y \right]_0^k - \int_0^k \tanh y \, dy$ $= k \tanh k - \ln \cosh k$	<p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p>	<p>For substitution to obtain equivalent integral</p> <p>Correct so far</p> <p>For integration by parts (correct way round)</p> <p>Final answer</p>

**Q6, (Jun 2011, Q8)**

<p><b>(i)</b></p>	$x = \cosh^2 u \Rightarrow du = 2 \cosh u \sinh u du$ $\int \sqrt{\frac{x}{x-1}} dx = \int \frac{\cosh u}{\sinh u} 2 \cosh u \sinh u du$ $= \int 2 \cosh^2 u du$ $= \int (\cosh 2u + 1) du = \sinh u \cosh u + u$ $= x^{\frac{1}{2}} (x-1)^{\frac{1}{2}} + \ln \left( x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}} \right) (+c)$	<p><b>B1</b></p> <p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p> <p><b>7</b></p>	<p>For correct result</p> <p>For substituting throughout for <math>x</math></p> <p>For correct simplified <math>u</math> integral</p> <p>For attempt to integrate <math>\cosh^2 u</math></p> <p>For correct integration</p> <p>For substituting for <math>u</math></p> <p>For correct result</p> <p><b>oe</b> as <math>f(x) + \ln(g(x))</math></p>
<p><b>(ii)</b></p>	$2\sqrt{3} + \ln(2 + \sqrt{3})$	<p><b>B1</b></p> <p><b>1</b></p>	
<p><b>(iii)</b></p>	$V = (\pi) \int_1^4 \frac{x}{x-1} dx = (\pi) [x + \ln(x-1)]_1^4$ $V \rightarrow \infty$	<p><b>M1</b></p> <p><b>A1</b></p> <p><b>B1</b></p> <p><b>3</b></p>	<p>For attempt to find <math>\int \frac{x}{x-1} dx</math></p> <p>For correct integration (ignore <math>\pi</math>)</p> <p>For statement that volume is infinite (independent of M mark)</p>



**Q8, (Jun 2012, Q1)**

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{(2x-1)^2 + 4} dx \text{ OR } \frac{1}{4} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\left(x-\frac{1}{2}\right)^2 + 1} dx$$

$$= \frac{1}{2} \left[ \frac{1}{2} \tan^{-1} \frac{2x-1}{2} \right]_{\frac{1}{2}}^{\frac{3}{2}} \text{ OR } \frac{1}{4} \left[ \tan^{-1} \left( x - \frac{1}{2} \right) \right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{4} \left( \tan^{-1} 1 - \tan^{-1} 0 \right) = \frac{1}{16} \pi$$

B1 For correct denominator (in 2nd case must include  $\frac{1}{4}$ )

M1 For integration to  $k \tan^{-1}(ax+b)$   
or  $k \ln \left( \frac{ax+b-c}{ax+b+c} \right)$

A1 FT for  $ax+b$  from their denominator  
For correct integration

M1 For substituting limits in any  $\tan^{-1}$  expression

A1 For correct value

**[5]**

**Q9, (Jan 2013, Q6)**

$x^2 + 4x + 8 = (x + 2)^2 + 4$ $\int_0^1 \frac{1}{\sqrt{x^2 + 4x + 8}} dx = \int_0^1 \frac{1}{\sqrt{(x + 2)^2 + 4}} dx$ $= \left[ \sinh^{-1} \frac{x + 2}{2} \right]_0^1 = \sinh^{-1} \left( \frac{3}{2} \right) - \sinh^{-1} 1$ $= \ln \left( \frac{3}{2} + \sqrt{1 + \frac{9}{4}} \right) - \ln(1 + \sqrt{2}) = \ln \left( \frac{3}{2} + \sqrt{\frac{13}{4}} \right) - \ln(1 + \sqrt{2})$ $= \ln \left( \frac{3 + \sqrt{13}}{2 + 2\sqrt{2}} \right)$	<p>M1 A1 M1 A1 M1 A1</p>	<p>Complete the square in order to use standard form  Use correct standard form in integration  Answer in <math>\sinh^{-1}</math> form  Attempt to turn into log form  www isw</p>
<p>Alternative for last 4 marks</p> $\int_0^1 \frac{1}{\sqrt{(x + 2)^2 + 4}} dx = \left[ \ln \left( (x + 2) + \sqrt{(x + 2)^2 + 4} \right) \right]_0^1$ $= \ln(3 + \sqrt{13}) - \ln(2 + \sqrt{8}) = \ln \left( \frac{3 + \sqrt{13}}{2 + 2\sqrt{2}} \right)$	<p>[6] M1 A1 M1 A1</p>	<p>Attempt to use Standard form  Limits  www isw</p>
<p>Alternative for last 4 marks</p> $x + 2 = 2 \tan \theta \Rightarrow I = \left[ \ln(\sec \theta + \tan \theta) \right]_{\pi/4}^{\tan^{-1} 3/2}$ $= \ln \left( \frac{3}{2} + \frac{\sqrt{13}}{2} \right) - \ln(1 + \sqrt{2}) = \ln \left( \frac{3 + \sqrt{13}}{2 + 2\sqrt{2}} \right)$	<p>M1 A1 M1 A1</p>	<p>Substitution Indefinite integral Deal with limits www isw</p>



**Q10, (Jun 2014, Q1)**

$\int_0^2 \frac{1}{\sqrt{4+x^2}} dx = \left[ \sinh^{-1} \left( \frac{x}{2} \right) \right]_0^2$ $= \sinh^{-1} 1 - \sinh^{-1} 0$ $= \ln 1 + \sqrt{1+1} - 0$ $= \ln 1 + \sqrt{2} \quad \text{cao} \quad \text{isw}$	<p><b>M1</b></p> <p><b>M1</b></p> <p><b>A1</b></p>	<p>Standard form</p> <p>Use of log form and substitute limits dep on 1st M</p>	
<p><b>Alternative:</b></p> $\int_0^2 \frac{1}{\sqrt{4+x^2}} dx = \left[ \ln x + \sqrt{x^2+4} \right]_0^2$ $= \ln 2 + \sqrt{8} - \ln 2$ $= \ln 1 + \sqrt{2}$	<p><b>[3]</b></p> <p><b>M1</b></p> <p><b>M1</b></p> <p><b>A1</b></p>	<p>Standard form</p> <p>Substitute limits</p>	

**Q11, (Jun 2015, Q3)**

$\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{2x-x^2}} dx = \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1+2x-x^2-1}} dx$ $= \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1-(1-x)^2}} dx$ $= \left[ -\sin^{-1}(1-x) \right]_{\frac{1}{2}}^1$ $= -\left( 0 - \frac{\pi}{6} \right) = \frac{\pi}{6}$	<p><b>M1</b></p> <p><b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p> <p><b>A1</b></p>	<p>Completing the square on given function</p> <p>By substitution or using standard form where completed square is of form <math>1-(1\pm x)^2</math></p> <p>Correct result of integration.</p> <p>Ignore limits</p> <p>Or</p> $= \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1-(x-1)^2}} dx$ $= \left[ \sin^{-1}(x-1) \right]_{\frac{1}{2}}^1 = \left( 0 - -\frac{\pi}{6} \right) = \frac{\pi}{6}$
<p><b>5</b></p>		

**Q12, (Jun 2017, Q2)**

	$I = \int_3^4 \frac{1}{x^2 - 6x + 10} dx = \int_3^4 \frac{1}{(x-3)^2 + 1} dx$ <p>From formula book:</p> $I = \left[ \tan^{-1}(x-3) \right]_3^4 = \tan^{-1} 1 \pm \tan^{-1} 0$ $= \frac{\pi}{4}$	<p><b>M1</b> <b>A1</b></p> <p><b>M1</b></p> <p><b>A1</b></p>	<p>Complete the square. Denominator</p> <p>Use standard form and apply limits</p> <p>www</p>	<p>Sight of <math>(x \pm 3)^2 + 1</math></p> <p>Accept <math>\tan^{-1}(x \pm 3)</math></p> <p>SC correct answer only B1</p>
		<p><b>[4]</b></p>		