First Order Differential Equations Involving Substitutions (From OCR 4727)

Q1, (Jun 2008, Q3)

(i)	$\frac{dz}{dx} = 1 + \frac{dy}{dx}$	M1		For differentiating substitution (seen or implied)
	$\frac{dz}{dx} - 1 = \frac{z+3}{z-1} \Rightarrow \frac{dz}{dx} = \frac{2z+2}{z-1} = \frac{2(z+1)}{z-1}$	A1		For correct equation in z **AEF**
		A1	3	For correct simplification to **AG**
(ii)	$\int \frac{z-1}{z+1} dz = 2\int dx$	B1		For $\int \frac{z-1}{z+1}(dz)$ and $\int (1)\,(dx)$ seen or implied
	$\Rightarrow \int 1 - \frac{2}{z+1} dz$ *OR* $\int 1 - \frac{2}{u} du = 2x\ (+c)$	M1		For rearrangement of LHS into integrable form *OR* substitution e.g. $u = z+1$ or $u = z-1$
	$\Rightarrow$ $z - 2\ln(z+1)$ *OR* $z + 1 - 2\ln(z+1)$ $= 2x\ (+c)$	A1		For correct integration of LHS as f(z)
	$\Rightarrow -2\ln(x+y+1) = x - y + c$	A1	4	For correct general solution **AEF**

Q2, (Jan 2009, Q5)

(i)	$y = u - \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{1}{x^2}$	M1		For differentiating substitution
		A1		For correct expression
	$x^3\left(\frac{du}{dx} + \frac{1}{x^2}\right) = x\left(u - \frac{1}{x}\right) + x + 1$	M1		For substituting y and $\frac{dy}{dx}$ into DE
	$\Rightarrow x^2 \frac{du}{dx} = u$	A1	4	For obtaining correct equation **AG**
(ii)	METHOD 1			
	$\int \frac{1}{u} du = \int \frac{1}{x^2} dx \Rightarrow \ln ku = -\frac{1}{x}$	M1		For separating variables and attempt at integration
		A1		For correct integration (k not required here)
	$ku = e^{-1/x} \Rightarrow k\left(y + \frac{1}{x}\right) = e^{-1/x}$	M1		For any 2 of } k seen, exponentiating, substituting for u
		M1		For all 3 of }
	$\Rightarrow y = Ae^{-1/x} - \frac{1}{x}$	A1	5	For correct solution **AEF** in form $y = \mathrm{f}(x)$
	METHOD 2			
	$\frac{du}{dx} - \frac{1}{x^2}u = 0 \Rightarrow$ I.F. $e^{\int -1/x^2 dx} = e^{1/x}$	M1		For attempt to find I.F.
	$\Rightarrow \frac{d}{dx}\left(u e^{1/x}\right) = 0$	A1		For correct result
	$u e^{1/x} = k \Rightarrow y + \frac{1}{x} = k e^{-1/x}$	M1		From $\boxed{u \times \text{I.F.} =}$, for k seen } in either order
		M1		for substituting for u }
	$\Rightarrow y = k e^{-1/x} - \frac{1}{x}$	A1		For correct solution **AEF** in form $y = \mathrm{f}(x)$

Q3, (Jun 2010, Q4)

	Working	Mark		Guidance
(i)	$y = xz \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$	B1		For correct differentiation of substitution
	$\Rightarrow xz + x^2\frac{dz}{dx} - xz = x\cos z \Rightarrow x\frac{dz}{dx} = \cos z$	M1 A1		For substituting into DE For DE in variables separable form
	$\Rightarrow \int \sec z\, dz = \int \frac{1}{x}dx$	M1		For attempt at integration to ln form on LHS
	$\Rightarrow \ln(\sec z + \tan z) = \ln kx$ *OR* $\ln\tan\left(\frac{1}{2}z + \frac{1}{4}\pi\right) = \ln kx$	A1		For correct integration (k not required here)
	$\Rightarrow \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = kx$ *OR* $\tan\left(\frac{y}{2x} + \frac{1}{4}\pi\right) = kx$	A1	6	For correct solution **AEF** including RHS $= e^{(\ln x)+c}$
(ii)	$(4, \pi) \Rightarrow \sec\frac{1}{4}\pi + \tan\frac{1}{4}\pi = 4k$ *OR* $\tan\left(\frac{1}{8}\pi + \frac{1}{4}\pi\right) = 4k$	M1		For substituting $(4, \pi)$ into their solution (with k)
	$\Rightarrow \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}\left(1 + \sqrt{2}\right)x$ *OR* $\tan\left(\frac{y}{2x} + \frac{1}{4}\pi\right) = \left(\frac{1}{4}\tan\frac{3}{8}\pi\right)x$ *or* $\frac{1}{4}\left(1+\sqrt{2}\right)x$	A1	2	For correct solution **AEF** Allow decimal equivalent $0.60355\,x$ Allow $e^{\ln x}$ for x
		8		

Q4, (Jun 2011, Q5)

	Working	Mark		Guidance
(i)	$\frac{dy}{dx} = ku^{k-1}\frac{du}{dx}$	M1 A1		For using chain rule For correct $\frac{dy}{dx}$
	$\Rightarrow xku^{k-1}\frac{du}{dx} + 3u^k = x^2u^{2k}$	M1		For substituting for y and $\frac{dy}{dx}$
	$\Rightarrow \frac{du}{dx} + \frac{3}{kx}u = \frac{1}{k}xu^{k+1}$	A1	4	For correct equation **AG**
(ii)	$k = -1$	B1	1	For correct k
(iii)	$\frac{du}{dx} - \frac{3}{x}u = -x \Rightarrow$ IF $e^{-\int\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$	B1√		For correct IF f.t. for IF = $x^{\frac{3}{k}}$ using k or their numerical value for k
	$\Rightarrow \frac{d}{dx}\left(u.\frac{1}{x^3}\right) = -\frac{1}{x^2}$	M1		For $\frac{d}{dx}(u.\text{their IF}) = -x.\text{their IF}$
	$\Rightarrow u.\frac{1}{x^3} = \frac{1}{x}(+c) \Rightarrow y = \frac{1}{cx^3 + x^2}$	A1 A1	4	For correct integration both sides For correct solution for y
		9		

Q5, (Jan 2012, Q1)

(i)	$(y = xu \Rightarrow)\ \frac{dy}{dx} = x\frac{du}{dx} + u$	B1	For a correct statement
	$x\frac{du}{dx} + u = \frac{2+u^2}{u}$	M1	For using the substitution to eliminate y (If B0, then y must be eliminated from LHS, but $\frac{d(uv)}{dx}$ sufficient)
	$\Rightarrow x\frac{du}{dx} = \frac{2}{u}$	A1	For correct equation **AG**
		[3]	
(ii)	$\int u\, du = \int \frac{2}{x}\, dx$	M1	For separating variables and writing/attempting integrals
	$\Rightarrow \frac{1}{2}u^2 = 2\ln((k)x)$ *OR* $\frac{1}{2}u^2 = 2\ln x\ (+c)$	A1	For correct integration both sides (k or c not required here)
	$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)^2 = 2\ln(kx)$ *OR* $\frac{1}{2}\left(\frac{y}{x}\right)^2 = 2\ln x + c$	M1	For substituting for u into integrated terms with constant (on either side)
	$\Rightarrow y^2 = 4x^2\ln(kx)$ *OR* $y^2 = 4x^2\ln x + Cx^2$	A1	For correct solution **AEF** $y^2 = f(x)$ Do not penalise "c" being used for different constants e.g. $2\ln x + c = 2\ln(cx)$
		[4]	

Q6, (Jun 2013, Q3)

$u = y^3 \Rightarrow \frac{du}{dx} = 3y^2\frac{dy}{dx}$	M1		Or $\frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}}\frac{du}{dx}$
in DE gives $x\frac{du}{dx} + 2u = \frac{\cos x}{x}$	A1		
$\frac{du}{dx} + \frac{2}{x}u = \frac{\cos x}{x^2}$	B1	Divide	Both sides
$I = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln x}$	M1	Correctly integrates	Must have form $\frac{du}{dx} + f(x)u = g(x)$
$= x^2$	A1		Can be implied by subsequent work
$x^2\frac{du}{dx} + 2xu = \cos x$			
$\frac{d}{dx}\left(x^2u\right) = \cos x$			
$x^2u = \sin x \quad (+A)$	M1	Integrate	
$u = \frac{\sin x + A}{x^2}$	A1	Or gives GS in implicit form	Must include constant at this stage
$y = \left(\frac{\sin x + A}{x^2}\right)^{\frac{1}{3}}$	A1		
	[8]		

Q7, (Jun 2014, Q2)

$u=y^2 \Rightarrow \frac{du}{dx}=2y\frac{dy}{dx}$	M1	Correctly finds	Or $\frac{dy}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}$
so DE $\Rightarrow 2y\frac{dy}{dx}-4y^2=2e^x$	M1	or for complete unsimplified substitution	
$\Rightarrow \frac{du}{dx}-4u=2e^x$	A1		Can be implied by next A1
$I=\exp\int -4\,dx=e^{-4x}$	A1ft		Must have form $\frac{du}{dx}+f(x)u=g(x)$ for this mark and any further marks Can be implied by subsequent work
$e^{-4x}\frac{du}{dx}-4e^{-4x}u=2e^{-3x}$	M1*	Multiples through by IF of form e^{kx}, simplifying RHS	
$ue^{-4x}=-\frac{2}{3}e^{-3x}(+A)$	*M1dep*	Integrates	
$u=-\frac{2}{3}e^x+Ae^{4x}$	M1dep *	Rearranges to make u or y^2 the subject	No more than 1 numerical error at this step
$y=\sqrt{-\frac{2}{3}e^x+Ae^{4x}}$	A1	Cao	ignore use of '±'
Alternative from 4th mark to 6th mark			
CF: (u=…) Ae^{4x}	A1		
PI: $u=ke^x$, $\frac{du}{dx}=ke^x$	M1*	PI chosen & differentiated correctly	
$ke^x-4ke^x=2e^x$, $k=-\frac{2}{3}$	M1 dep*	Substitutes and solves	
	[8]		

Q8, (Jun 2016, Q3)

(i)		$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$ $2u - xu^2\left(-u^{-2}\frac{du}{dx}\right) = \frac{1}{x^2}$ $x\frac{du}{dx} + 2u = \frac{1}{x^2}$	M1 M1 A1 [3]	Differentiate Substitute **ag** Convincingly shown	
(ii)		$\frac{du}{dx} + \frac{2u}{x} = \frac{1}{x^3}$ $I = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln x}$ $= x^2$ $\frac{d}{dx}(x^2u) = x^{-1}$ $x^2u = \ln x + A$ $u = (\ln x + A)/x^2$ $y = x^2/(\ln x + A)$ $x = 1, y = 1 \Rightarrow 1 = \frac{1}{0+A} \Rightarrow A = 1$ $y = \frac{x^2}{\ln x + 1}$	M1 A1 M1 A1 M1 M1 A1 [7]	$e^{k\ln x}$ for LHS, multiply and recognise derivative for y = reciprocal of 'their u' oe without fractions within fractions	incorrect IF means no further marks can be gained if RHS is not multiplied by IF then no further marks can be gained or ... = $\ln kx$ or $k = e$ or $y = \frac{x^2}{\ln ex}$