Differentiation of Inverse Trigonometric and Hyperbolic Functions (From OCR 4726)

Q1, (Jun 2006, Q2)

(i) Given that $y = \tan^{-1} x$, prove that $\frac{dy}{dx} = \frac{1}{1+x^2}$. [3]

(ii) Verify that $y = \tan^{-1} x$ satisfies the equation

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$ [3]

Q2, (Jun 2007, Q4)

(i) Given that

$$y = x\sqrt{1-x^2} - \cos^{-1} x,$$

find $\frac{dy}{dx}$ in a simplified form. [4]

(ii) Hence, or otherwise, find the exact value of $\int_0^1 2\sqrt{1-x^2}\,dx$. [3]

Q3, (Jan 2008, Q9i)

(i) Prove that $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$. [3]

Q4, (Jan 2010, Q9)

(i) Given that $y = \tanh^{-1} x$, for $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. **[3]**

(ii) It is given that $f(x) = a\cosh x - b\sinh x$, where a and b are positive constants.

(a) Given that $b \geqslant a$, show that the curve with equation $y = f(x)$ has no stationary points. **[3]**

(b) In the case where $a > 1$ and $b = 1$, show that $f(x)$ has a minimum value of $\sqrt{a^2-1}$. **[6]**

Q5, (Jun 2010, Q6)

(i) Show that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$. **[2]**

(ii) Given that $y = \cosh(a\sinh^{-1} x)$, where a is a constant, show that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2y = 0.$$ **[5]**

Q6, (Jun 2011, Q5)

(i) Prove that, if $y = \sin^{-1} x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. **[3]**

(ii) Find the Maclaurin series for $\sin^{-1} x$, up to and including the term in x^3. **[5]**

(iii) Use the result of part **(ii)** and the Maclaurin series for $\ln(1+x)$ to find the Maclaurin series for $(\sin^{-1} x)\ln(1+x)$, up to and including the term in x^4. **[4]**

Q7, (Jan 2012, Q6)

(i) Prove that the derivative of $\cos^{-1}x$ is $-\dfrac{1}{\sqrt{1-x^2}}$. **[3]**

A curve has equation $y = \cos^{-1}(1-x^2)$, for $0 < x < \sqrt{2}$.

(ii) Find and simplify $\dfrac{dy}{dx}$, and hence show that

$$\left(2-x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx}.$$

[5]

Q8, (Jun 2013, Q3)

It is given that $f(x) = \tanh^{-1}\left(\dfrac{1-x}{3+x}\right)$ for $x > -1$.

(i) Show that $f''(x) = \dfrac{1}{2(x+1)^2}$. **[6]**

(ii) Hence find the Maclaurin series for $f(x)$ up to and including the term in x^2. **[4]**

Q9, (Jun 2014, Q4)

The curves $y = \cos^{-1}x$ and $y = \tan^{-1}(\sqrt{2}x)$ intersect at a point A.

(i) Verify that the coordinates of A are $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{4}\pi\right)$. **[2]**

(ii) Determine whether the tangents to the curves at A are perpendicular. **[4]**

Q10, (Jun 2014, Q6)

(i) Given that $y = \cosh^{-1}x$, show that $y = \ln\left(x + \sqrt{x^2-1}\right)$. **[4]**

(ii) Show that $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2-1}}$. **[2]**

(iii) Solve the equation $\cosh x = 3$, giving your answers in logarithmic form. **[3]**

Q11, (Jun 2015, Q5)

It is given that $y = \sin^{-1}2x$.

(i) Using the derivative of $\sin^{-1}x$ given in the List of Formulae find $\dfrac{dy}{dx}$. **[1]**

(ii) Show that $(1-4x^2)\dfrac{d^2y}{dx^2} = 4x\dfrac{dy}{dx}$. **[3]**

(iii) Hence show that $(1-4x^2)\dfrac{d^3y}{dx^3} - 12x\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0$. **[2]**

(iv) Using your results from parts **(i)**, **(ii)** and **(iii)**, find the Maclaurin series for $\sin^{-1}2x$ up to and including the term in x^3. **[3]**
