Differentiation of Inverse Trigonometric and Hyperbolic Functions (From OCR 4726)

Q1, (Jun 2006, Q2)

(i)	Get $\sec^2 y \frac{dy}{dx} = 1$ or equivalent	M1	
	Clearly use $1 + \tan^2 y = \sec^2 y$	M1	May be implied
	Clearly arrive at A.G.	A1	
(ii)	Reasonable attempt to diff. to $\frac{-2x}{(1+x^2)^2}$	M1	Use of chain/quotient rule
	Substitute their expressions into D.E.	M1	Or attempt to derive diff. equ^n.
	Clearly arrive at A.G.	A1	
		SC	Attempt diff. of $(1+x^2)\frac{dy}{dx} = 1$ M1,A1 Clearly arrive at A.G. B1

Q2, (Jun 2007, Q4)

(i)	Reasonable attempt at product rule	M1	Two terms seen
	Derive or quote diff. of $\cos^{-1} x$	M1	Allow +
	Get $-x^2(1 - x^2)^{-½} + (1 - x^2)^{½} + (1 - x^2)^{-½}$	A1	
	Tidy to $2(1 - x^2)^{½}$	A1	cao
(ii)	Write down integral from (i)	B1	On any $k\sqrt{(1-x^2)}$
	Use limits correctly	M1	In any reasonable integral
	Tidy to ½π	A1	
		SR	Reasonable sub. B1 Replace for new variable and attempt to integrate (ignore limits) M1 Clearly get ½π A1

Q3, (Jan 2008, Q9i)

(i)	Get $\sinh y \, {}^{dy}/_{dx} = 1$	M1	Or equivalent; allow ± Allow use of ln equivalent with Chain Rule
	Replace $\sinh y = \sqrt{(\cosh^2 y - 1)}$	A1	
	Justify positive grad. to A.G.	B1	e.g. sketch

Q4, (Jan 2010, Q9)

(i)	Rewrite tanh y as $(e^y - e^{-y})/(e^y + e^{-y})$	B1	Or equivalent
	Attempt to write as quadratic in e^{2y}	M1	
	Clearly get A.G.	A1	
(ii)	**(a)** Attempt to diff. and solve = 0	M1	
	Get tanh $x = b/a$	A1	
	Use $(-1) < \tanh x < 1$ to show $b < a$	B1	
			SC Use exponentials — M1
			Get $e^{2x} = (a + b)/(a - b)$ — A1
			Use $e^{2x} > 0$ to show $b < a$ — B1
			SC Write $x = \tanh^{-1}(b/a)$ — M1
			$= \frac{1}{2}\ln((1 + b/a)/(1 - b/a))$ — A1
			Use ( ) > 0 to show $b < a$ — B1
	(b) Get tanh $x = 1/a$ from part **(ii)(a)**	B1	
	Replace as ln from their answer	M1	
	Get $x = \frac{1}{2}\ln((a + 1)/(a - 1))$	A1	
	Use $e^{\frac{1}{2}\ln((a+1)/(a-1))} = \sqrt{((a + 1)/(a - 1))}$	M1	At least once
	Clearly get A.G.	A1	
	Test for minimum correctly	B1	
			SC Use of $y = \cosh x(a - \tanh x)$ and $\cosh x = 1/\text{sech} x = 1/\sqrt{(1 - \tanh^2 x)}$

Q5, (Jun 2010, Q6)

(i)	Reasonable attempt to differentiate sinh $y = x$ to get dy/dx in terms of y	M1	Allow $\pm\cosh y\, dy/dx = 1$
	Replace sinh y to A.G.	A1	Clearly use $\cosh^2 - \sinh^2 = 1$
			SC Attempt to diff. $y = \ln(x+\sqrt{(x^2+1)})$ using chain rule — M1
			Clearly tidy to A.G. — A1
(ii)	Reasonable attempt at chain rule	M1	To give a product
	Get $dy/dx = a\sinh(a\sinh^{-1}x)/\sqrt{(x^2+1)}$	A1	
	Reasonable attempt at product/quotient	M1	Must involve sinh and cosh
	Get d^2y/dx^2 correctly in some form	A1√	From $dy/dx = k\sinh(a\sinh^{-1}x)/\sqrt{(x^2+1)}$
	Substitute in and clearly get A.G.	A1	
			SC Write $\sqrt{(x^2+1)}dy/dx = k\sinh(a\sinh^{-1}x)$ or similar
			Derive the A.G.

Q6, (Jun 2011, Q5)

(i)	$x = \sin y \Rightarrow \dfrac{dx}{dy} = \cos y$	**M1**	For implicit diffn to $\dfrac{dy}{dx} = \pm\dfrac{1}{\cos y}$ **oe**
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$	**A1**	For using $\sin^2 y + \cos^2 y = 1$ to obtain **N.B. Answer given**
	$+\sqrt{\ }$ taken since $\sin^{-1} x$ has positive gradient	**B1**	For justifying + sign
		3	
(ii)	$f(0) = 0,\ f'(0) = 1$	**B1**	For correct values
	$f''(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$	**M1**	Use of chain rule to differentiate $f'(x)$
	$f'''(x) = \dfrac{(1-x^2)^{\frac{3}{2}} + 3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$	**M1**	Use of quotient or product rule to differentiate $f''(0)$.
	$\Rightarrow f''(0) = 0,\ f'''(0) = 1$	**A1**	For correct values **www, soi**
	$\Rightarrow \sin^{-1} x = x + \frac{1}{6}x^3$	**A1** **5**	For correct series (allow 3!) **www**
	Alternative Method: $f(0) = 0, f'(0) = 1$	**B1**	For correct values
	$f'(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{3}{8}x^4 + \dots$	**M1**	Correct use of binomial
	$f''(x) = x + \dfrac{3}{2}x^3 + \dots$	**M1**	Differentiate twice
	$f'''(x) = 1 + \dfrac{9}{2}x^2 + \dots$		
	$\Rightarrow f'(0) = 1, f''(0) = 0, f'''(0) = 1$	**A1**	Correct values
	$\Rightarrow \sin^{-1} x = x + \frac{1}{6}x^3$	**A1**	Correct series
(iii)	$(\sin^{-1} x)\ln(1+x)$	**B1ft**	For terms in both series to at least x^3 f.t. from their **(ii)** multiplied together
	$= \left(x + \frac{1}{6}x^3\right)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)$	**M1**	For multiplying terms to at least x^3
	$= x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^4$	**A1**	For correct series up to x^3 **www**
		A1 **4**	For correct term in x^4 **www**

Q7, (Jan 2012, Q6)

(i)	$\cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$ $-$ sign since $\frac{dy}{dx} < 0$ (e.g. by graph)	M1 A1 B1 **[3]**	For differentiating $\cos y$ wrt x For using $\cos^2 y + \sin^2 y = 1$ to obtain **AG** For justification of $+\sqrt{\ }$ taken **SC1** if in fractions $\frac{14}{3}$ and $\frac{2047}{441}$	
(ii)	$\frac{dy}{dx} = -\frac{-2x}{\sqrt{1-(1-x^2)^2}}$ $= \frac{2x}{\sqrt{2x^2 - x^4}} = \frac{2}{\sqrt{2-x^2}}$ $\frac{d^2y}{dx^2} = 2.-\frac{1}{2}.-2x(2-x^2)^{-\frac{3}{2}} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$ $\Rightarrow (2-x^2)\frac{d^2y}{dx^2} = \frac{2x}{\sqrt{2-x^2}} = x\frac{dy}{dx}$	M1 A1 A1 M1 A1 **[5]**	For differentiating $\cos^{-1}(1-x^2)$ (as a function of a function) For correct $\frac{dy}{dx}$ (unsimplified) For correct $\frac{dy}{dx}$ (simplified) For differentiating $\frac{dy}{dx}$ using chain rule correctly (or product or quotient if y' is wrong) For verification of **AG**	

Q8, (Jun 2013, Q3)

(i)	$\frac{dy}{dx} = \frac{1}{1-\left(\frac{1-x}{3+x}\right)^2} \times \frac{-(3+x)-(1-x)}{(3+x)^2}$	B1 M1 A1	Sight of standard diffn for $\tanh^{-1} x$ Fn of fn **and** diffn of quotient Soi correct quotient (i.e. correct expression for 2nd part)
	$\Rightarrow \frac{dy}{dx} = \left(\frac{-4}{(3+x)^2-(1-x)^2}\right) = \frac{k}{1+x}$	A1	
	$\Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)}$	A1	Correct for y'
	$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+x)^2}$	A1 **[6]**	2^{nd} diffn (NB AG)
(ii)	When $x=0$, $y = \tanh^{-1}\frac{1}{3}$ or $\frac{1}{2}\ln 2$ or $\ln\sqrt{2}$	B1	For 1^{st} value (needs to be exact)
	$\frac{dy}{dx} = -\frac{1}{2}$ $\frac{d^2y}{dx^2} = \frac{1}{2}$	B1	For both
	$\Rightarrow y = \tanh^{-1}\frac{1}{3} + \left(-\frac{1}{2}\right)x + \left(\frac{1}{2}\right)\frac{x^2}{2}$	M1	Use of correct Maclaurin's series
	$= \tanh^{-1}\frac{1}{3} - \frac{1}{2}x + \frac{x^2}{4}$	A1 **[4]**	Accept 0.347

Q9, (Jun 2014, Q4)

(i)	For 1st curve $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$ For 2nd curve $\tan^{-1}\left(\sqrt{2}\times\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$	**B1** **B1** **[2]**		Alt: M1 Set up quadratic in sin or cos and solve A1 Both values correct
(ii)	For 1st curve $y=\cos^{-1}x$, $\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$	**B1**	**soi**	
	For 2nd curve $y=\tan^{-1}x$, $\frac{dy}{dx}=\frac{\sqrt{2}}{1+2x^2}$	**B1**	**soi**	
	For 1st curve, when $x=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}}=\frac{-1}{\frac{1}{\sqrt{2}}}=-\sqrt{2}$	**M1**	Substituting value into *their* derivatives and using $m_1\times m_2$ = ˜(1) (i.e. evidence of finding the product of gradients)	
	For 2nd curve, when $x=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$ Since $m_1\times m_2=-1$ then Yes	**A1** **[4]**	Depends on exact correct numerical values being seen	Acceptable reason: One the negative reciprocal of the other. Condone: One the negative inverse of the other

Q10, (Jun 2014, Q6)

(i)	$x = \cosh y = \dfrac{e^{y} + e^{-y}}{2} \Rightarrow e^{y} + e^{-y} = 2x$ $\Rightarrow e^{2y} - 2xe^{y} + 1 = 0$ $\Rightarrow e^{y} = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$ $\Rightarrow y = \ln\ x \pm \sqrt{x^2 - 1}$ Reject – sign as principal value taken $\Rightarrow y = \ln\ x + \sqrt{x^2 - 1}$	**M1** **A1** **B1** **A1** **[4]**	Finding 3 term quadratic in e^{y} Correct solution Including reason **oe**	Condone ignoring -ve sign at this point. Condone interchange of *x* and *y* but final ans must be correct
(ii)	$y = \ln\ x + \sqrt{x^2 - 1} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x + \sqrt{x^2 - 1}} \times \left(1 + \dfrac{2x}{2\sqrt{x^2 - 1}}\right)$ $= \dfrac{1}{x + \sqrt{x^2 - 1}} \times \dfrac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$	**M1** **A1** **[2]**	Alt: $x = \cosh y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sinh y}$ $= \dfrac{1}{\sqrt{x^2 - 1}}$	
(iii)	$x = \cosh^{-1} 3$ $= \ln\ 3 + \sqrt{8}$ $= -\ln\ 3 + \sqrt{8}$ **oe**	**M1** **A1** **A1** **[3]**	Use of $\cosh^{-1}$ **ft**, -ve the first answer	

Q11, (Jun 2015, Q5)

(i)		$y=\sin^{-1}(2x) \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-(2x)^2}}.\frac{d(2x)}{dx}$ $=\frac{2}{\sqrt{1-4x^2}}$	B1	Oe	
			1		
(ii)		$\frac{d^2y}{dx^2}=2\times\left(-\frac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x)=\frac{8x}{(1-4x^2)^{\frac{3}{2}}}$ $=\frac{8x}{(1-4x^2)\sqrt{1-4x^2}}=\frac{4x}{(1-4x^2)}\frac{dy}{dx}$ $(1-4x^2)\frac{d^2y}{dx^2}=4x\frac{dy}{dx}$	B1 M1 A1	For correct 2nd derivative Using *their* ans to connect 1st and 2nd derivatives Ft to achieve ag	**SC 2** if result obtained correctly from $y'=\frac{k}{\sqrt{(1-4x^2)}}$
			3		
(iii)		$(1-4x^2)\frac{d^3y}{dx^3}-8x\frac{d^2y}{dx^2}=4\frac{dy}{dx}+4x\frac{d^2y}{dx^2}$ $(1-4x^2)\frac{d^3y}{dx^3}-12x\frac{d^2y}{dx^2}-4\frac{dy}{dx}=0$	M1 A1	Using result of (ii) and product rule correctly	**M1** Starting with *their* 2nd derivative using appropriate method correctly **A1** ans www
			2		
(iv)		Find $y_0, y'_0, y''_0, y'''_0=\{0,2,0,8\}$ $y=0+2x+0+\frac{8x^3}{6}$ $\Rightarrow y=2x+\frac{4x^3}{3}$	B1 M1 A1	soi Correctly substituting *their* 4 values into correct Maclaurin www Ignore higher order terms	
			3		