DeMoivre's Theorem and Applications To Trigonometry (From OCR 4727)

Q1, (Jun 2007, Q5)

(i) $(\cos 6\theta =)\ \mathrm{Re}(c+\mathrm{i}s)^6$	M1		For expanding (real part of) $(c+\mathrm{i}s)^6$ at least 4 terms and 1 evaluated binomial coefficient needed
$(\cos 6\theta =)\ c^6-15c^4s^2+15c^2s^4-s^6$	A1		For correct expansion
$(\cos 6\theta =)$ $c^6-15c^4(1-c^2)+15c^2(1-c^2)^2-(1-c^2)^3$	M1		For using $s^2=1-c^2$
$(\cos 6\theta =)\ 32c^6-48c^4+18c^2-1$	A1	**4**	For correct result **AG**
(ii) $64x^6-96x^4+36x^2-3=0 \Rightarrow \cos 6\theta=\frac{1}{2}$	M1		For obtaining a numerical value of $\cos 6\theta$
$\Rightarrow (\theta =)\ \frac{1}{18}\pi, \frac{5}{18}\pi, \frac{7}{18}\pi$ etc.	A1		For any correct solution of $\cos 6\theta=\frac{1}{2}$
$\cos 6\theta=\frac{1}{2}$ has multiple roots	M1		For stating or implying at least 2 values of θ
largest x requires smallest θ $\Rightarrow$ largest positive root is $\cos\frac{1}{18}\pi$	A1	**4**	For identifying $\cos\frac{1}{18}\pi$ **AEF** as the largest positive root from a list of 3 positive roots *OR* from general solution *OR* from consideration of the cosine function
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Q2, (Jun 2008, Q4)

(i) $\cos^5\theta=\left(\dfrac{e^{i\theta}+e^{-i\theta}}{2}\right)^5$	B1		For $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$ seen or implied; z may be used for $e^{i\theta}$ throughout
$\cos^5\theta=\frac{1}{32}\left(e^{i\theta}+e^{-i\theta}\right)^5$	M1		For expanding $\left(e^{i\theta}+e^{-i\theta}\right)^5$. At least 3 terms and 2 binomial coefficients required *OR* reasonable attempt at expansion in stages
$\cos^5\theta=\frac{1}{32}\left(e^{5i\theta}+e^{-5i\theta}+5\left(e^{3i\theta}+e^{-3i\theta}\right)+10\left(e^{i\theta}+e^{-i\theta}\right)\right)$	A1		For correct binomial expansion
$\cos^5\theta=\frac{1}{16}(\cos 5\theta+5\cos 3\theta+10\cos\theta)$	M1		For grouping terms and using multiple angles
	A1	**5**	For answer obtained correctly **AG**
(ii) $\cos\theta=16\cos^5\theta$	B1		For stating correct equation of degree 5 *OR* $1=16\cos^4\theta$ **AEF**
$\Rightarrow \cos\theta=0,\quad \cos\theta=\pm\frac{1}{2}$	M1		For obtaining at least one of the values of $\cos\theta$ from $\cos\theta=k\cos^5\theta$ *OR* from $1=k\cos^4\theta$
$\Rightarrow \theta=\frac{1}{2}\pi,\ \frac{1}{3}\pi,\ \frac{2}{3}\pi$	A1		A1 for any two correct values of θ
	A1	**4**	A1 for the 3rd value and no more in $0 \leqslant \theta \leqslant \pi$. Ignore values outside $0 \leqslant \theta \leqslant \pi$
		9	

Q3, (Jan 2009, Q8)

(i)	$\sin\theta = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)$	B1		z may be used for $e^{\mathrm{i}\theta}$ throughout For expression for $\sin\theta$ seen or implied
	$\sin^6\theta =$	M1		For expanding $\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)^6$ At least 4 terms and 3 binomial coefficients required.
	$-\frac{1}{64}\left(e^{6\mathrm{i}\theta} - 6e^{4\mathrm{i}\theta} + 15e^{2\mathrm{i}\theta} - 20 + 15e^{-2\mathrm{i}\theta} - 6e^{-4\mathrm{i}\theta} + e^{-6\mathrm{i}\theta}\right)$	A1		For correct expansion. Allow $\frac{\pm(\mathrm{i})}{64}(\ldots\ldots)$
	$= -\frac{1}{64}(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20)$	M1		For grouping terms and using multiple angles
	$\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$	A1	5	For answer obtained correctly **AG**
(ii)	$\cos^6\theta =$ *OR* $\sin^6\left(\frac{1}{2}\pi - \theta\right) =$	M1		For substituting $\left(\frac{1}{2}\pi - \theta\right)$ for θ throughout
	$-\frac{1}{32}(\cos(3\pi - 6\theta) - 6\cos(2\pi - 4\theta) + 15\cos(\pi - 2\theta) - 10)$	A1		For correct unsimplified expression
	$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$	A1	3	For correct expression with $\cos n\theta$ terms **AEF**
(iii)	$\int_0^{\frac{1}{4}\pi} \frac{1}{32}(-2\cos 6\theta - 30\cos 2\theta)\,d\theta$	B1√		For correct integral. f.t. from $\sin^6\theta - \cos^6\theta$
	$= -\frac{1}{16}\left[\frac{1}{6}\sin 6\theta + \frac{15}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$	M1 A1√		For integrating $\cos n\theta$, $\sin n\theta$ *or* $e^{\mathrm{i}n\theta}$ For correct integration. f.t. from integrand
	$= -\frac{11}{24}$	A1	4	For correct answer **WWW**
			12	

Q4, (Jan 2010, Q7)

(i)	$\cos 6\theta = 0 \Rightarrow 6\theta = k \times \frac{1}{2}\pi$	M1		For multiples of $\frac{1}{2}\pi$ seen or implied
	$\Rightarrow \theta = \frac{1}{12}\pi\{1, 3, 5, 7, 9, 11\}$	A1		A1 for any 3 correct
		A1	3	A1 for the rest, and no extras in $0 < \theta < \pi$
(ii)	METHOD 1			
	$\mathrm{Re}(c + \mathrm{i}s)^6 = \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	M1		For expanding $(c + \mathrm{i}s)^6$ at least 4 terms and 2 binomial coefficients needed
		A1		For 4 correct terms
	$\cos 6\theta = c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3$	M1		For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$	A1		For correct expression for $\cos 6\theta$
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1	5	For correct result **AG** (may be written down from correct $\cos 6\theta$)

METHOD 2

	METHOD 2		
	$\text{Re}(c+\text{i}\,s)^3 = \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$	M1	For expanding $(c+\text{i}\,s)^3$ at least 2 terms and 1 binomial coefficient needed
		A1	For 2 correct terms
	$\Rightarrow \cos 6\theta = \cos 2\theta\left(\cos^2 2\theta - 3\sin^2 2\theta\right)$	M1	For replacing θ by 2θ
	$\Rightarrow \cos 6\theta = \left(2\cos^2\theta - 1\right)\left(4\left(2\cos^2\theta - 1\right)^2 - 3\right)$	A1	For correct expression in $\cos\theta$ (unsimplified)
	$\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$	A1	For correct result **AG**
(iii)	METHOD 1		
	$\cos 6\theta = 0$	M1	For putting $\cos 6\theta = 0$
	$\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $16c^4 - 16c^2 + 1 = 0$ and $2c^2 - 1 = 0$	A1	For association of roots with quartic and quadratic
	But $\theta = \frac{1}{4}\pi, \frac{3}{4}\pi$ satisfy $2c^2 - 1 = 0$	B1	For correct association of roots with quadratic
	EITHER Product of 4 roots *OR* $c = \pm\frac{1}{2}\sqrt{2 \pm \sqrt{3}}$	M1	For using product of 4 roots *OR* for solving quartic
	$\Rightarrow \cos\frac{1}{12}\pi\cos\frac{5}{12}\pi\cos\frac{7}{12}\pi\cos\frac{11}{12}\pi = \frac{1}{16}$	A1 5	For correct value (may follow A0 and B0)

Q5, (Jun 2013, Q8)

(i)		$\cos 5\theta + \text{i}\sin 5\theta = \left(\cos\theta + \text{i}\sin\theta\right)^5$	B1	Or $\cos 5\theta = re\{\left(\cos\theta + \text{i}\sin\theta\right)^5\}$
		$= c^5 + 5\text{i}c^4 s - 10c^3 s^2 - 10\text{i}c^2 s^3 + 5cs^4 + \text{i}s^5$	M1	
		$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$	M1	Take real parts
		$= c^5 - 10c^3\left(1 - c^2\right) + 5c\left(1 - c^2\right)^2$	M1	
		$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$		
		$\cos 5\theta = 16c^5 - 20c^3 + 5c$	A1 [5]	AG
(ii)		Multiplying by x gives $16x^5 - 20x^3 + 5x = 0$		
		letting $x = \cos\alpha$ gives $\cos 5\alpha = 0$	M1	
		hence $5\alpha = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \frac{9}{2}\pi$	A1	
		$\alpha = \frac{1}{10}\pi, \frac{3}{10}\pi, \frac{5}{10}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi$		
		$\cos\frac{5}{10}\pi = 0$ which is not a root	A1	
		so roots $x = \cos\frac{1}{10}\pi, \cos\frac{3}{10}\pi, \cos\frac{7}{10}\pi, \cos\frac{9}{10}\pi$	A1 [4]	
(iii)		$16x^4 - 20x^2 + 5 = 0 \Leftrightarrow x^2 = \dfrac{20 \pm \sqrt{80}}{32}$	B1	
		cos decreases between 0 and π so $\cos\frac{1}{10}\pi$ is greatest root	M1	
		so $\cos\frac{1}{10}\pi = \sqrt{\dfrac{20 + \sqrt{80}}{32}} = \sqrt{\dfrac{5 + \sqrt{5}}{8}}$	A1 [3]	Dep on full marks in (ii)

Q6, (Jan 2013, Q7)

(i)	(a)	$e^{i\theta}+e^{2i\theta}+\ldots+e^{10i\theta}=\dfrac{e^{i\theta}\left(\left(e^{i\theta}\right)^{10}-1\right)}{e^{i\theta}-1}$ $=\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta}}$ $=\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}$	M1 A1 M1 A1 [4]	Sum of a GP AG
(i)	(b)	$\theta=2n\pi \Rightarrow \text{sum}=10$	B1 [1]	
(ii)		$\cos\theta+\cos 2\theta+\ldots+\cos 10\theta=\text{Re}\left(\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}\right)$ $=\dfrac{\text{Re}\left(-ie^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)\right)}{2\sin\left(\frac{1}{2}\theta\right)}=\dfrac{\text{Re}\left(-ie^{\frac{21}{2}i\theta}+ie^{\frac{1}{2}i\theta}\right)}{2\sin\left(\frac{1}{2}\theta\right)}$	M1 M1	Take real parts Manipulate expression
		$=\dfrac{\sin\left(\frac{21}{2}\theta\right)-\sin\left(\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}$ $=\dfrac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}-\dfrac{1}{2}$	A1 [3]	AG
(iii)		$\cos\frac{1}{11}\pi+\cos\frac{2}{11}\pi+\ldots+\cos\frac{10}{11}\pi=\dfrac{\sin\left(\frac{21}{22}\pi\right)}{2\sin\left(\frac{1}{22}\pi\right)}-\dfrac{1}{2}$ But $\sin\frac{21}{22}\pi=\sin\left(\pi-\frac{21}{22}\pi\right)=\sin\frac{1}{22}\pi$ So $\text{RHS}=\frac{1}{2}-\frac{1}{2}=0$, so $\frac{1}{11}\pi$ is a root Using $\sin(2\pi+x)=\sin x$ gives $2\pi+\frac{1}{2}\theta=\frac{21}{2}\theta \Rightarrow \theta=\frac{1}{5}\pi$	M1 M1 A1 A1 [4]	 AG

Q7, (Jun 2014, Q7)

(i)	$2\mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}$ $2\mathrm{i}\sin n\theta = \mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{-\mathrm{i}n\theta}$	B1	any equivalent form
	$(2\mathrm{i}\sin\theta)^5 = \left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)^5$ $= \mathrm{e}^{\mathrm{i}5\theta} - 5\mathrm{e}^{\mathrm{i}3\theta} + 10\mathrm{e}^{\mathrm{i}\theta} - 10\mathrm{e}^{-\mathrm{i}\theta} + 5\mathrm{e}^{-\mathrm{i}3\theta} - \mathrm{e}^{-\mathrm{i}5\theta}$	M1*	binomial expansion
	$32i\sin^5\theta = (e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})$	M1dep*	grouping terms
	$= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$ $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$	A1	AG
		[4]	
(ii)	$16\sin^5\theta - 10\sin\theta = \sin 5\theta - 5\sin 3\theta$	M1*	Attempts to eliminate sin5θ and sin3θ
	$16\sin^5\theta - 6\sin\theta = 0$	A1	
	$\sin\theta = 0, \pm\sqrt[4]{\frac{3}{8}}$	M1dep*	must have 3 values for sin θ
	$\theta = 0, \pm 0.899$	A1	
		[4]	

Q8, (Jun 2016, Q7)

(i)	$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$	B1	Use de Moivre
	$= \cos^6\theta + 6i\sin\theta\cos^5\theta - 15\sin^2\theta\cos^4\theta - 20i\sin^3\theta\cos^3\theta + 15\sin^4\theta\cos^2\theta + 6i\sin^5\theta\cos\theta - \sin^6\theta$	B1	All terms correct
	$\sin 6\theta = 6\sin\theta\cos^5\theta - 20\sin^3\theta\cos^3\theta + 6\sin^5\theta\cos\theta$	M1	Compare imaginary parts
	$= \cos\theta\,(6\sin\theta\,(1-\sin^2\theta)^2 - 20\sin^3\theta\,(1-\sin^2\theta) + 6\sin^5\theta)$	M1	Take out factor of $\cos\theta$ and give other factor in terms of $\sin\theta$ only
	$= \cos\theta\,(32\sin^5\theta - 32\sin^3\theta + 6\sin\theta)$	A1 [5]	**ag** Convincingly shown, having been explicit about taking imaginary parts
(ii)	$\frac{\sin 6\theta}{\sin 2\theta} = \frac{\cos\theta\,(32\sin^5\theta - 32\sin^3\theta + 6\sin\theta)}{2\sin\theta\cos\theta}$ $= 16\sin^4\theta - 16\sin^2\theta + 3$	M1	
	$= 4(2\sin^2\theta - 1)^2 - 1$	M1	Complete the square
		A1	
	$\therefore \frac{\sin 6\theta}{\sin 2\theta} \geq -1$	M1	deduces lower bound
	$0 \leq 2\sin^2\theta \leq 2$ $\therefore (2\sin^2\theta - 1)^2 \leq 1$	M1	deduces upper bound
	$\therefore 4(2\sin^2\theta - 1)^2 - 1 \leq 3$ $\Rightarrow -1 \leq \frac{\sin 6\theta}{\sin 2\theta} \leq 3$		**SC** if none of marks 2 to 5 (M1A1M1M1) gained then **SC M1A1** for any valid method of deducing upper bound, and similarly for lower bound
	But upper bound attained $\Rightarrow \sin^2\theta = 0$ or 1 $\Rightarrow \sin 2\theta = 0$	M1	Dep on showing valid method for UB<=3
	So $\sin 2\theta \neq 0 \Rightarrow -1 \leq \frac{\sin 6\theta}{\sin 2\theta} < 3$	A1 [7]	full convincing overall argument

Q9, (Jun 2017, Q7)

(i)	$2\cos\theta = e^{i\theta} + e^{-i\theta}$ $2^6\cos^6\theta = e^{6i\theta} + 6e^{4i\theta} + 15e^{2i\theta} + 20 + 15e^{-2i\theta} + 6e^{-4i\theta} + e^{-6i\theta}$	M1 A1	Expand $\left(e^{i\theta} + e^{-i\theta}\right)^6$
	$2^6\cos^6\theta = \left(e^{6i\theta} + e^{-6i\theta}\right) + (6e^{4i\theta} + 6e^{-4i\theta}) + (15e^{2i\theta} + 15e^{-2i\theta}) + 20$ $\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$	M1	for converting to multiple angles
	$\Rightarrow$ result	A1 [4]	Complete argument including pairing up of e.g. terms in z^4 and z^{-4}
(ii)	$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3$ $\Rightarrow \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 3 + 13\cos 2\theta + 10$	M1* A1	Use result from (i) Oe simplified form
	$\Rightarrow 32\cos^6\theta = 13(1 + \cos 2\theta)$	*M1dep	Use double angle identity
	$\Rightarrow 32\cos^6\theta = 13(2\cos^2\theta)$	A1	
	$\Rightarrow \cos\theta = 0$ or $\cos^4\theta = \frac{13}{16}$	A1	
	$\theta = \frac{1}{2}\pi, 0.319, 2.82$	[5]	