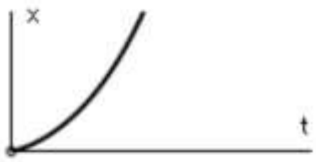
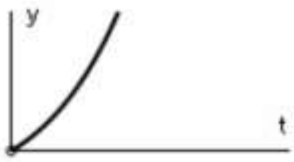
Coupled Simultaneous First Order Differential Equations (From OCR 4758)

Q1, (Jun 2007, Q4)

(i)	$\ddot{x} = -5\dot{x} + 4\dot{y} - 2e^{-2t}$	M1	Differentiate
	$= -5\dot{x} + 4\left(-9x + 7y + 3e^{-2t}\right) - 2e^{-2t}$	M1	Substitute for $\dot{y}$
	$= -5\dot{x} - 36x + \frac{28}{4}\left(\dot{x} + 5x - e^{-2t}\right) + 10e^{-2t}$	M1	y in terms of $x, \dot{x}$
		M1	Substitute for y
	$\ddot{x} - 2\dot{x} + x = 3e^{-2t}$	E1	
			5
(ii)	$\lambda^2 - 2\lambda + 1 = 0$	M1	Auxiliary equation
	$\lambda = 1$ (repeated)	A1	
	CF $x = (A + Bt)e^t$	F1	CF for their roots
	PI $x = ae^{-2t}$	B1	Correct form for PI
	$\dot{x} = -2ae^{-2t}, \ddot{x} = 4ae^{-2t}$	M1	Differentiate twice
	$4ae^{-2t} - 2\left(-2e^{-2t}\right) + ae^{-2t} = 3e^{-2t}$	M1	Substitute and compare
	$a = \frac{1}{3}$	A1	
	GS $x = \frac{1}{3}e^{-2t} + (A + Bt)e^t$	F1	GS = PI + CF (with two arbitrary constants)
			8
(iii)	$y = \frac{1}{4}\left(\dot{x} + 5x - e^{-2t}\right)$	M1	y in terms of $x, \dot{x}$
	$= \frac{1}{4}\left(-\frac{2}{3}e^{-2t} + Be^t + (A + Bt)e^t + \frac{5}{3}e^{-2t} + 5(A + Bt)e^t - e^{-2t}\right)$	M1	Differentiate x
		F1	$\dot{x}$ follows their x (but must use product rule)
	$y = \frac{1}{4}e^t(6A + B + 6Bt)$	A1	cao
			4
(iv)	$\frac{1}{3} + A = 0$	M1	Condition on x
	$\frac{1}{4}(6A + B) = 0$	M1	Condition on y
	$A = -\frac{1}{3}, B = 2$		
	$x = \frac{1}{3}e^{-2t} + \left(2t - \frac{1}{3}\right)e^t$	A1	Both solutions correct
	$y = 3te^t$		
	$t = 0 \Rightarrow \dot{x} = 1, \dot{y} = 3$	B1	Both values correct
	 	B1	x through origin and consistent with their solution for large t (but not linear)
		B1	y through origin and consistent with their solution for large t (but not linear)
		B1	Gradient of both curves at origin consistent with their values of $\dot{x}, \dot{y}$
			7

Q2, (Jun 2008, Q4)

(i)	$\ddot{x} = 4\dot{x} - 6\dot{y} - 9\cos t$	M1	Differentiate first equation	
	$= 4\dot{x} - 6(3x - 5y - 7\sin t) - 9\cos t$	M1	Substitute for $\dot{y}$	
	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = 4\dot{x} - 18x + 5(4x - \dot{x} - 9\sin t) + 42\sin t - 9\cos t$	M1	Substitute for y	
	$\ddot{x} + \dot{x} - 2x = -3\sin t - 9\cos t$	E1	LHS	
		E1	RHS	
				6
(ii)	$\alpha^2 + \alpha - 2 = 0$	M1	Auxiliary equation	
	$\alpha = 1$ or -2	A1		
	CF $x = Ae^t + Be^{-2t}$	F1	CF for their roots	
	PI $x = a\cos t + b\sin t$	B1	PI of this form	
	$(-ac - bs) + (-as + bc) - 2(ac + bs) = -3s - 9c$	M1	Differentiate twice and substitute	
	$-a + b - 2a = -9$	M1	Compare coefficients (2 equations)	
	$-b - a - 2b = -3$	M1	Solve (2 equations)	
	$\Rightarrow a = 3, b = 0$	A1		
	$x = 3\cos t + Ae^t + Be^{-2t}$	F1	Their PI + CF (with two arbitrary constants)	
				9
(iii)	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$= \frac{1}{6}\left(12\cos t + 4Ae^t + 4Be^{-2t} + 3\sin t - Ae^t + 2Be^{-2t} - 9\sin t\right)$	M1	Differentiate x and substitute	
	$y = 2\cos t - \sin t + \frac{1}{2}Ae^t + Be^{-2t}$	A1	Constants must correspond with those in x	
				3
(iv)	x bounded $\Rightarrow A = 0$	M1	Identify coefficient of exponentially growing term must be zero	
	$\Rightarrow y$ bounded	E1	Complete argument	
				2
(v)	$t = 0, y = 0 \Rightarrow 0 = B + 2 \Rightarrow B = -2$	M1	Condition on y	
	$x = 3\cos t - 2e^{-2t}$, $y = 2\cos t - \sin t - 2e^{-2t}$	F1	Follow their (non-trivial) general solutions	
	$x = 3\cos t$	A1	cao	
	$y = 2\cos t - \sin t$	A1	cao	
				4

Q3, (Jan 2011, Q4)

(i)		$y=10\dot{x}-x$ $\dot{y}=10\ddot{x}-\dot{x}$ $10\ddot{x}-\dot{x}=-0.2x+3\dot{x}-0.3x$ $\ddot{x}-0.4\dot{x}+0.05x=0$	M1 M1 M1 M1 E1 **[5]**	 Eliminate y Eliminate $\dot{y}$	
(ii)		$\lambda^2-0.4\lambda+0.05=0$ $\lambda=0.2\pm0.1\mathrm{j}$ $x=\mathrm{e}^{0.2t}(A\cos 0.1t+B\sin 0.1t)$	M1 A1 M1 F1 **[4]**	Auxiliary equation CF for complex roots CF for their roots	
(iii)		$\dot{x}=0.2\mathrm{e}^{0.2t}(A\cos 0.1t+B\sin 0.1t)$ $+0.1\mathrm{e}^{0.2t}(-A\sin 0.1t+B\cos 0.1t)$ $y=10\dot{x}-x$ $=\mathrm{e}^{0.2t}((A+B)\cos 0.1t+(B-A)\sin 0.1t)$	M1 A1 M1 A1 **[4]**	Differentiate (product rule) Substitute to find y	
(iv)		$x_0=A$ $y_0=A+B$ $x=\mathrm{e}^{0.2t}(x_0\cos 0.1t+(y_0-x_0)\sin 0.1t)$ $y=\mathrm{e}^{0.2t}(y_0\cos 0.1t+(y_0-2x_0)\sin 0.1t)$	B1 M1 A1 A1 **[4]**	 Use condition	
(v)		$y=0$ when $\tan 0.1t=\dfrac{-y_0}{y_0-2x_0}=-1.25$ So (for least positive t), $t=22.5$ $x=0$ when $\tan 0.1t=\dfrac{-x_0}{y_0-x_0}=-\dfrac{1}{9}$ So (for least positive t), $t=30.3$ Hence rabbits die out first	M1 F1 A1 M1 F1 A1 A1 **[7]**	 Or compare values of $\tan 0.1t$ Or compare values of $\tan 0.1t$ Complete argument	

Q4, (Jan 2012, Q4)

(i)	$y=\frac{1}{2}(\dot{x}+x)$	M1	Express y as subject
	$\dot{y}=\frac{1}{2}(\ddot{x}+\dot{x})$	M1	Express $\dot{y}$ as subject
	$\frac{1}{2}(\ddot{x}+\dot{x})=-x-\frac{4}{2}(\dot{x}+x)+e^{-2t}$	M1	Substitute for y
		M1	Substitute for $\dot{y}$
	$\ddot{x}+5\dot{x}+6x=2e^{-2t}$	A1	
	AE $\lambda^2+5\lambda+6=0$	M1	
	$\lambda=-2$ or -3	A1	FT their AE
	CF $Ae^{-2t}+Be^{-3t}$	F1	
	PI $x=ate^{-2t}$	B1	cao
	$\dot{x}=-2ate^{-2t}+ae^{-2t}$, $\ddot{x}=4ate^{-2t}-4ae^{-2t}$	M1	Differentiate twice
	$4ate^{-2t}-4ae^{-2t}+5\left(-2ate^{-2t}+ae^{-2t}\right)+6ate^{-2t}=2e^{-2t}$	M1	Substitute
	$-4a+5a=2$	M1	Compare
	$a=2$	A1	
	GS $x=2te^{-2t}+Ae^{-2t}+Be^{-3t}$	F1	PI + CF with 2 arbitrary constants
		[14]	
(ii)	$\frac{dx}{dt}=-4te^{-2t}+2e^{-2t}-2Ae^{-2t}-3Be^{-3t}$	M1	
	$y=\frac{1}{2}(\dot{x}+x)$	M1	Substitute for $x,\dot{x}$
	$y=(1-t)e^{-2t}-\frac{1}{2}Ae^{-2t}-Be^{-3t}$	A1	cao
		[3]	
(iii)	$t=0, x=5\Rightarrow 5=A+B$	M1	Use condition
	$t=0, y=0\Rightarrow 0=1-\frac{1}{2}A-B$	M1	Use condition
	A = 8, B = -3		
	$x=2te^{-2t}+8e^{-2t}-3e^{-3t}$	A1	cao
	$y=-te^{-2t}-3e^{-2t}+3e^{-3t}$	A1	cao
		[4]	

(iv)		$\frac{y}{x}=\frac{-t-3+3e^{-t}}{2t+8-3e^{-t}}\to-\frac{1}{2}$	E1	Convincingly shown
		$\frac{y}{x}=-\frac{1}{2}\Leftrightarrow -t-3+3e^{-t}=-t-4+\frac{3}{2}e^{-t}$		
		$\Leftrightarrow \frac{3}{2}e^{-t}=-1$	M1	Or attempt other valid argument e.g. $y\downarrow, x\uparrow$ so $y/x\downarrow$
		but $e^{-t}>0$ for any t, so no values of t	E1 **[3]**	Complete argument

Q5, (Jan 2013, Q4)

(i)		$y=\frac{2}{3}\left(-\dot{x}-\frac{1}{2}x+t\right)$	M1		
		$\dot{y}=\frac{2}{3}\left(-\ddot{x}-\frac{1}{2}\dot{x}+1\right)$	M1	Differentiate	
		$\frac{2}{3}\left(-\ddot{x}-\frac{1}{2}\dot{x}+1\right)=\frac{3}{2}x-\frac{1}{2}\cdot\frac{2}{3}\left(-\dot{x}-\frac{1}{2}x+t\right)+2t$	M1	Substitute	
		$2\ddot{x}+2\dot{x}+5x=2-5t$	A1	oe	
		AE $2\lambda^2+2\lambda+5=0$	M1		
		$\lambda=-\frac{1}{2}\pm\frac{3}{2}j$	A1		
		CF $e^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t+B\sin\frac{3}{2}t\right)$	M1	Correct form	
			F1	FT wrong roots	
		PI $x=a+bt$	B1		
		$\dot{x}=b,\ \ddot{x}=0\Rightarrow 2b+5(a+bt)=2-5t$	M1	Differentiate and substitute	
		$\left.\begin{array}{r}2b+5a=2\\5b=-5\end{array}\right\}\Rightarrow a=\frac{4}{5}, b=-1$	M1 A1	Equate coefficients and solve	
		GS $x=\frac{4}{5}-t+e^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t+B\sin\frac{3}{2}t\right)$	F1		
			[13]		

(ii)		$y=\frac{2}{3}\left(-\dot{x}-\frac{1}{2}x+t\right)$	M1		
		$\dot{x}=-1-\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos\frac{3}{2}t+B\sin\frac{3}{2}t\right)$	M1	Must be using product rule	
		$+e^{-\frac{1}{2}t}\left(-\frac{3}{2}A\sin\frac{3}{2}t+\frac{3}{2}B\cos\frac{3}{2}t\right)$	F1	Must be GS from **(i)**	
		$y=\frac{2}{5}+t+e^{-\frac{1}{2}t}\left(A\sin\frac{3}{2}t-B\cos\frac{3}{2}t\right)$	A1		
			[4]		
(iii)		$x=1,t=0\Rightarrow 1=\frac{4}{5}+A\Rightarrow A=\frac{1}{5}$	M1		
		$y=0,t=0\Rightarrow 0=\frac{2}{5}-B\Rightarrow B=\frac{2}{5}$	M1		
		$x=\frac{4}{5}-t+e^{-\frac{1}{2}t}\left(\frac{1}{5}\cos\frac{3}{2}t+\frac{2}{5}\sin\frac{3}{2}t\right)$			
		$y=\frac{2}{5}+t+e^{-\frac{1}{2}t}\left(\frac{1}{5}\sin\frac{3}{2}t-\frac{2}{5}\cos\frac{3}{2}t\right)$	A1	Both	
			[3]		
(iv)		$x+y=\frac{6}{5}+e^{-\frac{1}{2}t}\left(\frac{3}{5}\sin\frac{3}{2}t-\frac{1}{5}\cos\frac{3}{2}t\right)$	M1	Adding and attempting the limit	
		$t\to\infty\Rightarrow e^{-\frac{1}{2}t}\to 0\Rightarrow x+y\to\frac{6}{5}$	E1	FT for finite limit	
		$x+y=\frac{6}{5}\Leftrightarrow\frac{3}{5}\sin\frac{3}{2}t-\frac{1}{5}\cos\frac{3}{2}t=0\Leftrightarrow\tan\frac{3}{2}t=\frac{1}{3}$	M1	Establish equation and indicate method	
		which occurs (infinitely often)	E1	Correctly investigate the existence of a solution, but explicit solution for t not required.	
			[4]		

Q6, (Jun 2015, Q4)

(i)		$\frac{d^2x}{dt^2}=2\frac{dx}{dt}-2\frac{dy}{dt}$	M1	Differentiate	
		$=2\frac{dx}{dt}-\frac{3}{4}x+60e^{-\frac{1}{2}t}$	M1	Substitute for $\frac{dy}{dt}$	
		$\frac{d^2x}{dt^2}-2\frac{dx}{dt}+\frac{3}{4}x=60e^{-\frac{1}{2}t}$	A1	AG Rearrange	
		Auxiliary equation $m^2-2m+\frac{3}{4}=0$	M1		
		$m=\frac{1}{2},\frac{3}{2}$	A1		
		CF: $x=Ae^{\frac{3}{2}t}+Be^{\frac{1}{2}t}$	F1		
		PI: $x=Pe^{-\frac{1}{2}t}$	B1	Correct form for their CF	
		$x'=-\frac{1}{2}Pe^{-\frac{1}{2}t}, x''=\frac{1}{4}Pe^{-\frac{1}{2}t}$	M1	Differentiate and substitute	
		$P=30$	A1	Solve	
		$x=Ae^{\frac{3}{2}t}+Be^{\frac{1}{2}t}+30e^{-\frac{1}{2}t}$	F1	PI + CF with 2 arb const	
			[10]		
(ii)		$y=x-\frac{1}{2}\frac{dx}{dt}$	M1	Rearrange	
		Substitute for x and $\frac{dx}{dt}$	M1		
		$y=\frac{1}{4}Ae^{\frac{3}{2}t}+\frac{3}{4}Be^{\frac{1}{2}t}+\frac{75}{2}e^{-\frac{1}{2}t}$	A1	cao. As final answer	
			[3]		

(iii)		$x=40, t=0 \Rightarrow 40=A+B+30$ $y=50, t=0 \Rightarrow 50=\frac{1}{4}A+\frac{3}{4}B+\frac{75}{2}$ $A=-10, B=20$ $x=-10e^{\frac{3}{2}t}+20e^{\frac{1}{2}t}+30e^{-\frac{1}{2}t}$ $y=-\frac{5}{2}e^{\frac{3}{2}t}+15e^{\frac{1}{2}t}+\frac{75}{2}e^{-\frac{1}{2}t}$	M1 M1 A1 A1 **[4]**	Use condition Use condition cao cao	
(iv)		When $x=0$ $\quad 20e^{T}-10e^{2T}+30=0$ $e^{2T}-2e^{T}-3=0$ $e^{T}=3$ (or -1) $T=\ln 3\,(=1.10)$ $y=34.64$	M1 M1 A1 A1 M1 A1 **[6]**	Multiply by $e^{\frac{1}{2}T}$ Attempt to solve as a quadratic cao Substitute T in expression for y cao	
(v)		Unsuitable, X is negative	B1 **[1]**		

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Q7, (Jun 2017, Q4)

(i)	$\ddot{x}=2\dot{x}+2\dot{y}$ $\ddot{x}=2\dot{x}+2(6y-4x)$ $2y=\dot{x}-2x$ $\ddot{x}-8\dot{x}+20x=0$ AE $m^2-8m+20=0$ $m=4\pm 2\text{i}$ CF $x=\text{e}^{4t}(A\cos 2t+B\sin 2t)$	M1 M1 M1 A1 M1 A1 F1 **[7]**	Differentiate Substitute for $\dot{y}$ Substitute for y oe
(ii)	$\dot{x}=\text{e}^{-4t}(-2A\sin 2t+2B\cos 2t+4A\cos 2t+4B\sin 2t)$ Use $2y=\dot{x}-2x$ Substitute x and $\dot{x}$ $y=e^{4t}((A+B)\cos 2t+(B-A)\sin 2t)$	M1 M1 M1 A1 **[4]**	Differentiate x as a product Substitute into expression for y Simplify
(iii)	$t=0, \dot{x}=10$: $2B+4A=10$ $A+B=kA$ $x=\text{e}^{4t}(\frac{5}{k+1}\cos 2t+\frac{5(k-1)}{k+1}\sin 2t)$ $y=\text{e}^{4t}(\frac{5k}{k+1}\cos 2t+\frac{5(k-2)}{k+1}\sin 2t)$	M1 M1 M1 A1 A1 **[5]**	Use condition Use condition $y=kx$ when $t=0$ Solve equations
(iv)	A species dies out when x or $y=0$ $\cos 2t+5\sin 2t=0$ or $6\cos 2t+4\sin 2t=0$ $\tan 2t=-\frac{1}{5}$ or $\tan 2t=-\frac{6}{4}$ Solves one of these equations: $t=1.47$ or $t=1.08$ Species Y dies out first, $t=1.08$	B1 M1 A1 M1 A1 A1 F1 **[7]**	 Uses $k=6$ to obtain one of these equations Only one expression required Second value of t Correct conclusion for their values of t
(v)	Model does not hold once one species has died out	B1 **[1]**	

Q8, (Jun 2018, Q4)

(i)	$y''+9x'-y'=7e^{7t}$	M1	Differentiate	
	$y''+9(7x+2y+13e^{4t})-y'=7e^{7t}$	M1	Substitute for x'	
	$x=\frac{1}{9}\left(y-y'+e^{7t}\right)$	M1	Rearrange	
	$y''+7(y-y'+e^{7t})+18y+117e^{4t})-y'=7e^{7t}$	M1	Substitute for x	
	$y''-8y'+25y=-117e^{4t}$	A1		
	Auxiliary equation: $m^2-8m+25=0$	M1		
	$m=4\pm 3i$	A1		
	CF: $y=e^{4t}\left(A\cos 3t+B\sin 3t\right)$	F1		
	PI: $y=Pe^{4t}$	B1	Correct form for their RHS	
	$y'=4Pe^{4t}, y''=16Pe^{4t}$	M1	Differentiate, substitute and solve	
	$A=-13$	A1	cao	
	GS: $y=e^{4t}\left(A\cos 3t+B\sin 3t\right)-13e^{4t}$	F1	CF with 2 arb constants + PI	
		[12]		
	Alternative scheme for finding differential equation in x:			
	$\ddot{x}=7\dot{x}+2\dot{y}+52e^{4t}$	M1	Differentiate	
	$\ddot{x}=7\dot{x}+2(-9x+y+e^{7t})+52e^{4t}$	M1	Substitute for y'	
		M1	Rearrange and substitute for y	
	$\ddot{x}-8\dot{x}+25x=39e^{4t}+2e^{7t}$	A1	cao	
	Auxiliary eqn: $m^2-8m+25=0$	M1		
	$m=4\pm 3i$	A1		
	CF: $x=e^{4t}(D\cos 3t+E\sin 3t)$	F1		
	PI: $x=Pe^{4t}+Qe^{7t}$	M1	Correct form for their RHS	
	$P=\frac{13}{3},\ Q=\frac{1}{9}$	A1	cao	
	$x=e^{4t}(D\cos 3t+E\sin 3t)+\frac{13}{3}e^{4t}+\frac{1}{9}e^{7t}$			
	Differentiate	M1		
	Substitute and rearrange to find y	M1		
	$y=\frac{1}{2}e^{4t}(A\cos 3t+B\sin 3t)-13e^{4t}$	A1	cao	

(ii)	$y=-3, t=0$: $A=10$ $y' = e^{4t}(-3A\sin 3t + 3B\cos 3t) - 52e^{4t} + 4e^{4t}(A\cos 3t + B\sin 2t)$ $y' = 60, t=0$: $B=24$ $y = e^{4t}(10\cos 3t + 24\sin 3t) - 13e^{4t}$	M1 M1 M1 A1 **[4]**	Use condition Differentiate using product rule Use condition cao	
(iii)	Substitute for y and y' in $x = \frac{1}{9}(y - y' + e^{7t})$ $x = \frac{1}{9}(e^{7t} + e^{4t}(-102\cos 3t - 42\sin 3t + 39))$	M1 A1 **[2]**	cao oe (must be simplified)	
(iv)	$(10\cos 3t + 24\sin 3t) = 13$ $\sin(3t+\alpha) = \frac{1}{2}$ or $\cos(3t-\alpha) = \frac{1}{2}$ where $\tan\alpha = \frac{5}{12}$ ($\alpha = 0.394$) $t = \frac{\pi}{18} - \frac{1}{3}\tan^{-1}\left(\frac{5}{12}\right)$	M1 M1 A1 A1 **[4]**	Equate y to zero and attempt to solve Write in correct form or any other appropriate method $\tan\alpha = \frac{12}{5}$ if using cos formula, ($\alpha = 1.176$) or 0.0429	
(v)	Divide y and x by e^{7t} or e^{4t} and consider limit as $t \to \infty$ Limit is 0 AG	M1 E1 **[2]**		